

ALGEBRAIC GEOMETRY — EXERCISE SHEET 1
DUE ON 29/11/2023

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Exercise 0.1. Prove that $\mathbb{A}_{\mathbb{F}}^1$ is infinite for any field \mathbb{F} .

Exercise 0.2. Exhibit an isomorphism of schemes $\mathrm{Spec} \mathbf{k}[x, y]/(y - x^2) \cong \mathbb{A}_{\mathbf{k}}^1$. Show that there *cannot* be an isomorphism $\mathrm{Spec} \mathbf{k}[x, y]/(x^2 + y^2 - 1) \cong \mathbb{A}_{\mathbf{k}}^1$.

Exercise 0.3. Prove that a scheme X is connected if and only if $\mathcal{O}_X(X)$ has only the trivial idempotents 0, 1.

Exercise 0.4. Let $A \hookrightarrow \mathbf{k}[t]$ be \mathbf{k} -the subalgebra generated by t^2 and t^3 .

- (1) Prove that the \mathbf{k} -algebra homomorphism

$$\mathbf{k}[x, y] \xrightarrow{\pi} A, \quad x \mapsto t^2, \quad y \mapsto t^3$$

is surjective and induces an isomorphism $\mathbf{k}[x, y]/(x^3 - y^2) \xrightarrow{\sim} A$.

- (2) Consider the inclusion $\phi: A \hookrightarrow \mathbf{k}[t]$. Show that the induced morphism of schemes $f_{\phi}: \mathbb{A}_{\mathbf{k}}^1 \rightarrow \mathrm{Spec} A$ is bijective on points, but not an isomorphism.

Exercise 0.5. Let (X, \mathcal{O}_X) be a scheme, $U \subset X$ an open subset. Prove that $(U, \mathcal{O}_X|_U)$ is a scheme.

Exercise 0.6. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Set $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Prove that $\mathrm{Frac}(A/\mathfrak{p}) = \kappa(\mathfrak{p})$.

Exercise 0.7. Decide whether the following affine schemes are irreducible (resp. connected):

- (1) $\mathrm{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$,
- (2) $\mathrm{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$,
- (3) $\mathrm{Spec} \mathbb{C}[x, y, z]/(x^2 - yz, xz - x)$,
- (4) $\mathrm{Spec} \mathbb{Z}[x]/(2x)$,
- (5) $\mathrm{Spec} \mathbb{C}[x, y]/(xy, y^2)$,
- (6) $\mathrm{Spec} \mathbb{C}[x, y]/(x^2, xy, y^3)$,
- (7) $\mathrm{Spec}(A \times A')$, where A and A' are rings,
- (8) $\mathrm{Spec} \mathbb{C}[x, y, z]/(xy - z^2)$,
- (9) $\mathrm{Spec} \mathbb{C}[x, y]/(x^2 + y^2 - 1)$.

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