ALGEBRAIC GEOMETRY — EXERCISE SHEET 2
DUE ON 21/10/2023

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Exercise 0.1. Let $X = \operatorname{Spec} A$ be an irreducible affine variety of dimension d over a field $\mathbf{k}$. Prove (granting Krull's theorem if you like) that, for any $f \in A \setminus \sqrt{0}$, the closed subset $\mathrm{V}(f) \subset X$ is of pure dimension $d-1$.

Exercise 0.2. Find examples where $\operatorname{Proj} B$ is an integral scheme but B is not an integral domain.

Exercise 0.3. Let $f\colon X \to Y$ be a morphism of integral schemes with generic points $\xi_X \in X$ and ξ_Y. Show that the following are equivalent.

(1) $f(X) \subset Y$ is dense,
(2) $f^{\#}\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ is injective,
(3) for every open $V \subset Y$ and every open $U \subset f^{-1}(V)$, the map $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective,
(4) $f(\xi_X) = \xi_Y$,
(5) $\xi_Y \in \operatorname{im}(f)$.

If these conditions are satisfies, we say that f is *dominant*.

Exercise 0.4. Let X be a noetherian scheme. Show that the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is reduced (resp. is an integral domain) is open.

Exercise 0.5. Set $X = \operatorname{Proj} \mathbf{k}[x,y,z,w]/(xz-y^2, yz-xw, z^2-yw)$. Compute the field of rational functions on X, and deduce that $\dim X = 1$.

Exercise 0.6. Prove that the only fat points over $\mathbb{C}$ of length 3, up to isomorphism of $\mathbb{C}$-schemes, are $\operatorname{Spec} \mathbb{C}[t]/t^3$ and $\operatorname{Spec} \mathbb{C}[x,y]/(x^2,xy,y^2)$. [1]

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[1] Recall: a fat point over $\mathbb{C}$ is a scheme of the form $\operatorname{Spec} A$ for A a local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Its length is $\dim_{\mathbb{C}} A$.