ALGEBRAIC GEOMETRY — EXERCISE SHEET 1
DUE ON 25/10/2024

Exercise 0.1. Prove that $\mathbb{A}^1_{\mathbb{F}}$ is infinite for any field $\mathbb{F}$.

Exercise 0.2. Exhibit an isomorphism of schemes $\operatorname{Spec} \mathbf{k}[x,y]/(y-x^2) \cong \mathbb{A}^1_{\mathbf{k}}$. Show that there *cannot* be an isomorphism $\operatorname{Spec} \mathbf{k}[x,y]/(x^2+y^2-1) \cong \mathbb{A}^1_{\mathbf{k}}$.

Exercise 0.3. Prove that a scheme X is connected if and only if $\mathscr{O}_X(X)$ has only the trivial idempotents $0,1$.

Exercise 0.4. Let $A \hookrightarrow \mathbf{k}[t]$ be $\mathbf{k}$-the subalgebra generated by t^2 and t^3.

(1) Prove that the $\mathbf{k}$-algebra homomorphism

$$\mathbf{k}[x,y] \xrightarrow{\pi} A, \quad x \mapsto t^2, \; y \mapsto t^3$$

is surjective and induces an isomorphism $\mathbf{k}[x,y]/(x^3-y^2) \xrightarrow{\sim} A$.

(2) Consider the inclusion $\phi \colon A \hookrightarrow \mathbf{k}[t]$. Show that the induced morphism of schemes $f_\phi \colon \mathbb{A}^1_{\mathbf{k}} \to \operatorname{Spec} A$ is bijective on points, but not an isomorphism.

Exercise 0.5. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Set $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Prove that $\operatorname{Frac}(A/\mathfrak{p}) = \kappa(\mathfrak{p})$.

Exercise 0.6. Decide whether the following affine schemes are irreducible (resp. connected):

(1) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2-x^2(x+1))$,
(2) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2-x^3)$,
(3) $\operatorname{Spec} \mathbb{C}[x,y,z]/(x^2-yz, xz-x)$,
(4) $\operatorname{Spec} \mathbb{Z}[x]/(2x)$,
(5) $\operatorname{Spec} \mathbb{C}[x,y]/(xy, y^2)$,
(6) $\operatorname{Spec} \mathbb{C}[x,y]/(x^2, xy, y^3)$,
(7) $\operatorname{Spec}(A \times A')$, where A and A' are rings,
(8) $\operatorname{Spec} \mathbb{C}[x,y,z]/(xy-z^2)$,
(9) $\operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2-1)$.

Exercise 0.7. Let A be a ring. Prove that $\mathscr{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) = A$.

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