ALGEBRAIC GEOMETRY — EXERCISE SHEET 3
DUE ON 17/11/2023

Exercise 0.1. Let X be a noetherian scheme. Show that the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is reduced (resp. an integral domain) is open.

Exercise 0.2. Let $X = \operatorname{Spec} A$ be an irreducible affine variety of dimension d over a field $\mathbf{k}$. Prove (granting Krull's theorem if you like) that, for any $f \in A \setminus \sqrt{0}$, the closed subset $\mathrm{V}(f) \subset X$ is of pure dimension $d-1$.

Exercise 0.3. Find examples where $\operatorname{Proj} B$ is an integral scheme but B is not an integral domain.

Exercise 0.4. Let $f: X \to Y$ be a morphism of integral schemes with generic points $\xi_X \in X$ and $\xi_Y \in Y$. Show that the following are equivalent.

(1) $f(X) \subset Y$ is dense,
(2) $f^{\#}: \mathscr{O}_Y \to f_* \mathscr{O}_X$ is injective,
(3) for every open $V \subset Y$ and every open $U \subset f^{-1}(V)$, the map $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is injective,
(4) $f(\xi_X) = \xi_Y$,
(5) $\xi_Y \in \operatorname{im}(f)$.

If these conditions are satisfied, we say that f is *dominant*.

Exercise 0.5. Let A be a ring. Prove that sending $\mathfrak{p} \mapsto \mathfrak{p}^{\mathrm{h}}$ defines a morphism of schemes $\mathbb{A}^{n+1}_A \setminus \mathrm{V}(x_0, \ldots, x_n) \to \mathbb{P}^n_A$.

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