ALGEBRAIC GEOMETRY — EXERCISE SHEET 4

Exercise 0.1. Construct a $\mathbb{C}$-scheme $X \subset \mathbb{A}^2_{\mathbb{C}}$ such that $X_{\mathrm{red}} = \operatorname{Spec} \mathbb{C}[x,y]/(y-x^2)$ and X is nonreduced at the point $(x-1, y-1)$ only.

Exercise 0.2. Let B be a graded A-algebra. Prove the following.

(i) If B is noetherian, then $\operatorname{Proj} B$ is noetherian.
(ii) If B is finitely generated over A, then $\operatorname{Proj} B \to \operatorname{Spec} A$ is of finite type.
(iii) If B is a domain, then $\operatorname{Proj} B$ is integral.
(iv) If B is reduced, then $\operatorname{Proj} B$ is reduced.

Exercise 0.3. Let $\phi\colon B \to C$ be a graded homomorphism of graded A-algebras. Set $U = \operatorname{Proj} C \setminus \mathrm{V}_+(B_+C)$. Show that if $\phi_d\colon B_d \xrightarrow{\sim} C_d$ for $d \gg 0$, then $U = \operatorname{Proj} C$ and the induced morphism $\operatorname{Proj} C \to \operatorname{Proj} B$ is an isomorphism.

Exercise 0.4. Confirm that the affine cone $\pi\colon X = \operatorname{Spec} \mathbf{k}[x,y,z]/(x^2 - yz) \to \operatorname{Spec} \mathbf{k}$ is singular using the infinitesimal lifting criterion.

Exercise 0.5. Let $X \hookrightarrow \mathbb{P}^n_A$ be a closed immersion. Show that there exists a homogeneous ideal $I \subset B = A[x_0, x_1, \ldots, x_n]$ such that X is isomorphic to $\operatorname{Proj} B/I$. Show, by exhibiting an example, that I is not unique with this property.

Exercise 0.6. Prove that if $X \to S$ is surjective and $T \to S$ is an arbitrary morphism, then $X \times_S T \to T$ is surjective. Prove that this fails with surjective replaced by injective or bijective.

Exercise 0.7. Let $\mathbb{F}$ be a field. Find examples of connected (resp. irreducible, resp. integral) $\mathbb{F}$-schemes X such that $X \times_{\mathbb{F}} \mathbb{E}$ is not connected (resp. irreducible, resp. integral) for some finite extension $\mathbb{E} \supset \mathbb{F}$.

Exercise 0.8. Let $X \to S$ be a morphism of schemes, $s \in S$ a point. Show that $|X_s|$ agrees with the set-theoretic fibre $f^{-1}(s) \subset X$.

Exercise 0.9. Let X be a $\mathbf{k}$-scheme locally of finite type. Show that the set of closed points is dense.

Exercise 0.10. Show that, for any positive integer $n \geq 1$, the projective variety $X = \operatorname{Proj} \mathbf{k}[x,y,z]/(x^n + y^n + z^n) \subset \mathbb{P}^2_{\mathbf{k}}$ is regular if and only if $\gcd(\mathsf{char}\, \mathbf{k}, n) = 1$.

Exercise 0.11. Show that the map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is not flat as soon as $n \geq 2$. Show that the algebra homomorphism $\mathbf{k}[t^2, t^3] \hookrightarrow \mathbf{k}[t]$ is not flat.

Exercise 0.12. Show that if $f\colon X \to S$ is faithfully flat the maps $f^\#_x\colon \mathscr{O}_{S,s} \to \mathscr{O}_{X,x}$ are injective.

Exercise 0.13. Let A be a reduced ring. Then the set of zero-divisors in A is the union of the minimal primes of A.

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