Algebraic Geometry

Andrea T. Ricolfi

SISSA

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[...] *Oscar Zariski bewitched me. When he spoke the words "algebraic variety", there was a certain resonance in his voice that said distinctly that he was looking into a secret garden. I immediately wanted to be able to do this too. It led me to 25 years of struggling to make this world tangible and visible.*

David Mumford

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0 | Before we start

About this course

This is a 50 hours course (2.5 cycles for SISSA students).

The exam consists of an oral presentation by the student about a topic mutually agreed on, plus a few questions regarding the material covered in the course.

For UNITS students: the exam can only be scheduled within the official exam session, see the academic calendar.

Prerequisites

Familiarity with basic theory of commutative rings and modules is of great help, but not necessary. The relevant notions will be recalled as we need them. We have, however, included Appendix B to cover the basic commutative algebra constructions we will be referring to (and much more), and Appendix A to cover the basics of category theory as well.

Conventions

We list here a series of conventions that will be used throughout this text.

- The axiom of choice (or Zorn's Lemma) is assumed; so, for instance, every ring has a maximal ideal, and a poset $(P, \leq)$ in which every chain has an upper bound admits a maximal element.
- Given two sets A and B, the phrase '$A \subset B$' means that A is contained in B, *possibly equal* to B.
- A *ring* is a commutative, unitary ring. The zero ring (the one where $1 = 0$) is allowed (and in fact needed), but we always assume our rings are nonzero unless we explicitly mention it. Ring homomorphisms preserve the identity.
- By $\mathbf{k}$ we indicate an algebraically closed field, by $\mathbb{F}$ an arbitrary field.

- An open cover of a topological space U is the datum of a set I, and an open subset $U_i \subset U$ for every $i \in I$, such that $U = \bigcup_{i\in I} U_i$. We set $U_{ij} = U_i \cap U_j$. If $I = \emptyset$, then $U = \emptyset$.

- To say that Ω is an object a category $\mathscr{C}$ we simply write '$\Omega \in \mathscr{C}$' instead of $\Omega \in \mathrm{Ob}(\mathscr{C})$, with the exception of Appendix A, where a crash course on categories and functors is provided.

Main references

We list here a series of bibliographical references that integrate this text.

- Q. Liu, *Algebraic geometry and arithmetic curves* [11],
- R. Hartshorne, *Algebraic geometry* [8],
- R. Vakil, *The rising sea* [17],
- D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry* [5],
- M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra* [1],

1 | Introduction

Algebraic Geometry deals with the study of *algebraic varieties*. At a first approximation, these are common zero loci of collections of polynomials, i.e. solutions to systems

$$\begin{cases} f_1(x_1,\ldots,x_n)=0 \\ \quad\vdots \\ f_r(x_1,\ldots,x_n)=0 \end{cases}$$

of polynomial equations. When $\deg f_j = 1$ for all $j=1,\ldots,r$, this is the content of *Linear Algebra*, but the higher degree case poses nontrivial difficulties!

The concept of algebraic variety has been vastly generalised by Grothendieck's theory of *schemes*, introduced in [7].

Figure 1.1: Alexander Grothendieck (1928–2014).

This course is an introduction to schemes and to (part of) the massive dictionary, shared by all algebraic geometers, centered around schemes. Even though algebraic varieties are somewhat 'easier' objects, schemes are an incredibly useful and powerful tool to study them.

In this introduction, we briefly recap the key relation

$$\text{Algebra} \longleftrightarrow \text{Geometry}$$

in the land of *classical* algebraic varieties. We provide no proofs for now, but you shouldn't worry about this, because we will be proving more general results in the main body of these notes.

Let $\mathbf{k}$ be an algebraically closed field. Classical *affine n-space over* $\mathbf{k}$ is just

$$\mathbb{A}^n_{\mathbf{k}} = \{ (a_1, \ldots, a_n) \mid a_i \in \mathbf{k} \text{ for } i = 1, \ldots, n \}.$$

We denote it $\mathbb{A}^n_{\mathbf{k}}$ and not $\mathbf{k}^n$ to emphasise that we view it as a set of points rather than a vector space over $\mathbf{k}$. For instance, $\mathbb{A}^1_{\mathbf{k}}$ is called the *affine line* over $\mathbf{k}$, and $\mathbb{A}^2_{\mathbf{k}}$ is called the *affine plane* over $\mathbf{k}$. Let

$$A = \mathbf{k}[x_1, \ldots, x_n]$$

be the polynomial ring in n variables over the field $\mathbf{k}$. Each element $f \in A$ defines a function $\widetilde{f} \colon \mathbb{A}^n_{\mathbf{k}} \to \mathbf{k}$ sending $(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$, and since $\mathbf{k}$ is algebraically closed one has $f = g$ if and only if $\widetilde{f} = \widetilde{g}$.[1] Thus we shall just write f instead of $\widetilde{f}$.

Let $I = (f_1, \ldots, f_r) \subset A$ be an arbitrary ideal (here we are using that every ideal in A is finitely generated, by Hilbert's basis theorem [9]). The 'vanishing locus'

$$\mathrm{V}(I) = \left\{ (a_1, \ldots, a_n) \in \mathbb{A}^n_{\mathbf{k}} \,\middle|\, f_j(a_1, \ldots, a_n) = 0 \text{ for } j = 1, \ldots, r \right\} \subset \mathbb{A}^n_{\mathbf{k}}$$

is called an *algebraic set*. There is precisely one topology on $\mathbb{A}^n_{\mathbf{k}}$ having the algebraic sets as closed sets. It is called the *Zariski topology*.

Indeed, one has

- $\mathbb{A}^n_{\mathbf{k}} = \mathrm{V}(0)$,
- $\emptyset = \mathrm{V}(A)$,
- $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(IJ)$,
- $\bigcap_{s \in S} \mathrm{V}(I_s) = \mathrm{V}\big(\sum_{s \in S} I_s\big)$ for any family of ideals $(I_s \subset A)_{s \in S}$.

Figure 1.2: Oscar Zariski (1899–1986).

Example 1.0.1. Every ideal in $\mathbf{k}[x]$ is principal, i.e. of the form (f) for some $f \in \mathbf{k}[x]$. Since $\mathbf{k}$ is algebraically closed, we have $f = \alpha(x - a_1) \cdots (x - a_d)$, for $\alpha, a_1, \ldots, a_d \in \mathbf{k}$, and where $d = \deg f$. Thus, if $f \neq 0$, then $\mathrm{V}(f) = \{a_1, \ldots, a_d\} \subset \mathbb{A}^1_{\mathbf{k}}$, proving that all proper closed subsets of $\mathbb{A}^1_{\mathbf{k}}$ are finite. In particular, all open sets are infinite (since $\mathbf{k}$ is algebraically closed, thus infinite).

[1] For instance, the field $\mathbb{F}_3 = \{0, 1, 2\}$ is not algebraically closed, and the polynomials $f = x^2 + 1$ and $g = x^4 + 1$ are different, nevertheless one has $\widetilde{f} = \widetilde{g}$ as functions on the three point space $\mathbb{A}^1_{\mathbb{F}_3}$.

We have thus established an assignment

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\ldots,x_n]\} \xrightarrow{\mathrm{V}(-)} \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\}.$$

Conversely, given a subset $S \subset \mathbb{A}^n_{\mathbf{k}}$, the assignment

$$\mathrm{I}(S) = \left\{ f \in A \,\middle|\, f(p) = 0 \text{ for all } p \in S \right\} \subset A$$

defines a map the other way around, namely

$$\{\text{ideals } I \subset \mathbf{k}[x_1,\ldots,x_n]\} \xleftarrow{\mathrm{I}(-)} \{\text{subsets } S \subset \mathbb{A}^n_{\mathbf{k}}\}.$$

The two maps are *not* inverse to each other, even if we restrict I(−) to algebraic sets. For instance, consider the ideal $(x^r) \subset \mathbf{k}[x]$ for $r > 1$. Then $\mathrm{V}(x^r) = \{0\}$, and thus $\mathrm{I}(\mathrm{V}(x^r)) = (x)$, which is strictly larger than (x^r). The next result says that this is what *always* happens.

THEOREM 1.0.2 (Hilbert's Nullstellensatz [10]). *Let $I \subset \mathbf{k}[x_1,\ldots,x_n]$ be an ideal, where $\mathbf{k}$ is an algebraically closed field. Then,* $\mathrm{I}(\mathrm{V}(I)) = \sqrt{I}$*, i.e. $f \in \mathrm{I}(\mathrm{V}(I))$ if and only if $f^r \in I$ for some $r > 0$.*

See [11, Ch. 2, Corollary 1.15] for a modern proof of Hilbert's Nullstellensatz.

Composing our two assignments the other way around, we also find something larger than what we started with: consider for instance the complement $S \subset \mathbb{A}^1_{\mathbf{k}}$ of a finite set. Then $\mathrm{I}(S) = (0)$, since there are no nonzero polynomials with infinitely many zeroes. Thus $\mathrm{V}(\mathrm{I}(S)) = \mathbb{A}^1_{\mathbf{k}}$. In general, if S is an arbitrary subset of $\mathbb{A}^n_{\mathbf{k}}$, one can easily prove the identity

$$\mathrm{V}(\mathrm{I}(S)) = \overline{S},$$

where $\overline{S}$ is the closure of S in $\mathbb{A}^n_{\mathbf{k}}$ (with respect to the Zariski topology), namely the smallest algebraic set containing S. Thus in order to get $\mathrm{V}(\mathrm{I}(S)) = S$ we have to start with an algebraic set S (which is closed by definition).

Furthermore, one can prove that an algebraic set $Y \subset \mathbb{A}^n_{\mathbf{k}}$ is irreducible (i.e. it cannot be written as a union of two proper closed subsets) if and only if $\mathrm{I}(Y) \subset A$ is a prime ideal.

An irreducible algebraic set in $\mathbb{A}^n_{\mathbf{k}}$ is called an *affine variety in* $\mathbb{A}^n_{\mathbf{k}}$.

Of course, an affine variety carries the induced Zariski topology by default. Combining these observations together, we obtain correspondences (with 'algebra' on the left, and 'geometry' on the right)

$$\begin{array}{ccc}
\{\text{radical ideals in } \mathbf{k}[x_1,\ldots,x_n]\} & \underset{\mathrm{I}(-)}{\overset{\mathrm{V}(-)}{\rightleftarrows}} & \{\text{algebraic sets in } \mathbb{A}^n_{\mathbf{k}}\} \\
\uparrow & & \uparrow \\
\{\text{prime ideals in } \mathbf{k}[x_1,\ldots,x_n]\} & \underset{\mathrm{I}(-)}{\overset{\mathrm{V}(-)}{\rightleftarrows}} & \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\}
\end{array}$$

where an ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$ is *radical* if $I = \sqrt{I}$ (Definition 3.1.2).

Recall that, by definition, a *finitely generated* **k**-*algebra* is a **k**-algebra B isomorphic to a quotient $\mathbf{k}[x_1, \ldots, x_n]/I$ for some n and some ideal $I \subset \mathbf{k}[x_1, \ldots, x_n]$. Such a B is an integral domain (i.e. as a ring it has no nonzero zero-divisors) precisely when I is prime. Thus the bottom correspondence above can be rephrased as

$$\{\, \mathbf{k}[x_1, \ldots, x_n]/\mathfrak{p} \mid \mathfrak{p} \text{ is prime} \,\} \underset{\mathrm{I}(-)}{\overset{\mathrm{V}(-)}{\rightleftarrows}} \{\text{affine varieties in } \mathbb{A}^n_{\mathbf{k}}\,\}.$$

In the first part of this course, we will extend this correspondence to arbitrary *rings* on the left. What will be constructed on the right will be called an *affine scheme*, and what we shall establish is not just a bijection, but an equivalence of categories

$$\mathsf{Rings}^{\mathrm{op}} \cong \text{Affine schemes.}$$

Affine schemes are the basic building blocks for the construction of general *schemes*, in the same way as open subsets of $\mathbb{R}^m$ are the basic building blocks for m-dimensional smooth manifolds. As we shall see, a *scheme* is defined by the property that every point has an open neighborhood isomorphic to an affine scheme.

2 | Sheaves

Sheaves were defined by Leray (1906–1998), while he was a prisoner in Austria during World War II.

Sheaves are a key notion present in the toolbox of every mathematician keen to understand the "nature" of a *geometric space*. They incarnate one of the basic principles that will be unraveled throughout this course, which can be stated as the slogan

$$\boxed{\text{geometric spaces are determined by functions on them.}}$$

Even though there may be "few" functions on a space X, a complete knowledge of all functions on all open subsets of X allows one, in principle, to reconstruct X. This local-to-global principle is perfectly encoded in the notion of a sheaf.

2.1 Key example: smooth functions

Before diving into precise definitions, we explore a key example of sheaf.

Let X be a smooth manifold. For each open subset $U \subset X$, we have a ring (actually, an $\mathbb{R}$-algebra)

$$C^\infty(U,\mathbb{R}) = \{\text{smooth functions } U \to \mathbb{R}\}.$$

Indeed, smooth functions with the same source can naturally be added and multiplied exploiting the ring structure on $\mathbb{R}$. If $V \hookrightarrow U$ is an open subset, we have a restriction map

$$\rho_{UV} \colon C^\infty(U,\mathbb{R}) \to C^\infty(V,\mathbb{R}), \quad f \mapsto f|_V,$$

which is an $\mathbb{R}$-algebra homomorphism. One has $\rho_{UU} = \mathrm{id}_{C^\infty(U,\mathbb{R})}$, and if $W \hookrightarrow V \hookrightarrow U$ is a chain of open subsets of X, we have a commutative diagram

$$C^\infty(U,\mathbb{R}) \xrightarrow{\rho_{UV}} C^\infty(V,\mathbb{R}) \xrightarrow{\rho_{VW}} C^\infty(W,\mathbb{R}), \qquad \rho_{UW} = \rho_{VW} \circ \rho_{UV}.$$

So far, we have just observed that the assignment $U \mapsto C^\infty(U,\mathbb{R})$ is *functorial*, from open subsets of X (which form a category) to the category of $\mathbb{R}$-algebras. The two distinguished features of the assignment $U \mapsto C^\infty(U,\mathbb{R})$, which make it into a *sheaf* of $\mathbb{R}$-algebras on X, are the following:

(i) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i\in I} U_i$. If $f, g \in C^\infty(U,\mathbb{R})$ are smooth functions such that $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then $f = g$. In other words, a smooth function is determined by its restriction to the open subsets forming a covering. This is the *locality axiom*.

(ii) Fix an open subset $U \subset X$ and an open cover $U = \bigcup_{i\in I} U_i$. Given a smooth function $f_i \in C^\infty(U_i,\mathbb{R})$ on each U_i, such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for every $(i, j) \in I \times I$, there is a smooth function $f \in C^\infty(U,\mathbb{R})$ such that $f_i = f|_{U_i}$ for every $i \in I$. In other words, functions glue along an open cover. This is the *glueing axiom*.

A sheaf is an abstract notion formalising this "ability of glueing". The formal definition will be given in Important Definition 2.2.1. Note that the result of the glueing in Condition (ii) is *unique* by Condition (i).

Let us continue with our example. Let $x \in X$ be a point. Consider the ring

$$C^\infty_{X,x} = \left\{ (U, f) \,\middle|\, x \in U,\ f \in C^\infty(U,\mathbb{R}) \right\} \big/ \sim$$

where $(U, f) \sim (V, g)$ whenever there exists an open subset $W \subset U \cap V$, containing x, such that $f|_W = g|_W$. Note that $C^\infty_{X,x}$ is indeed a ring, with addition and multiplication

$$[U, f] + [U', f'] = [U \cap U', f + f']$$
$$[U, f] \cdot [U', f'] = [U \cap U', f f'].$$

This ring, which is in fact an $\mathbb{R}$-algebra via $c \mapsto [X, c]$, for all $c \in \mathbb{R}$, is called the *stalk* of the sheaf $C^\infty(-,\mathbb{R})$ at x (cf. Important Definition 2.4.1), and it receives a natural $\mathbb{R}$-algebra homomorphism from $C^\infty(U,\mathbb{R})$ for every open subset U of X such that $x \in U$, sending $f \mapsto [U, f]$. The image of f along this map is called the *germ of f at x*. The subset

$$\mathfrak{m}_x = \left\{ [U, f] \in C^\infty_{X,x} \,\middle|\, f(x) = 0 \right\} \subset C^\infty_{X,x}$$

forms an ideal, which is a *maximal* ideal, being the kernel of the (surjective) evaluation map

$$\begin{aligned} C^\infty_{X,x} &\twoheadrightarrow \mathbb{R} \\ [U, f] &\longmapsto f(x). \end{aligned}$$

In fact, $\mathfrak{m}_x$ is the *unique* maximal ideal of $C^\infty_{X,x}$. To see this, it is enough to check that every element of $C^\infty_{X,x} \setminus \mathfrak{m}_x$ is invertible. But this is true, since a smooth function that is nonzero in a neighbourhood of x is invertible there.

The upshot is, then, that the pair $(C^\infty_{X,x}, \mathfrak{m}_x)$ defines a *local ring* with residue field $\mathbb{R}$. The geometric spaces X one deals with in algebraic geometry, namely *schemes*, have precisely this property: they come with a sheaf of rings $\mathscr{O}_X$ such that each stalk $\mathscr{O}_{X,x}$ is a local ring. These spaces $(X, \mathscr{O}_X)$ actually form a larger category, that of locally ringed spaces (cf. Section 2.10). Schemes are particular instances of locally ringed spaces.

2.2 Presheaves, sheaves, morphisms

Let $\mathscr{C}$ be a concrete category (Definition A.1.16) with a final object $0 \in \mathscr{C}$. The concreteness assumption means that part of the structure is the datum of a faithful functor $F : \mathscr{C} \to \mathsf{Sets}$, but we will (for the moment) ignore this datum. To fix ideas, $\mathscr{C}$ should be thought of as any of the following categories:

- $\mathscr{C} = \mathsf{Sets}$,
- $\mathscr{C} = \mathsf{Rings}$,
- $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$,
- $\mathscr{C} = \mathsf{Mod}_R$, where R is a ring.

If X is a topological space, we denote by τ_X the category of open subsets of X. The set $\mathrm{Hom}_{\tau_X}(V, U)$ between two open sets $V, U \subset X$ is just the empty set if $V \not\subset U$, or the singleton $\{ V \hookrightarrow U \}$ in case V is contained in U. Thus the opposite category τ_X^{op} satisfies

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) = \begin{cases} \{ V \hookrightarrow U \} & \text{if } V \subset U \\ \emptyset & \text{if } V \not\subset U \end{cases}$$

and a functor $\mathcal{F} : \tau_X^{\mathrm{op}} \to \mathscr{C}$ (i.e. a contravariant functor $\tau_X \to \mathscr{C}$) determines a map

$$\mathrm{Hom}_{\tau_X^{\mathrm{op}}}(U, V) \to \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V)),$$

which is nothing but a choice of an element $\rho_{UV} \in \mathrm{Hom}_{\mathscr{C}}(\mathcal{F}(U), \mathcal{F}(V))$ for any inclusion of open subsets $V \subset U$.

Definition 2.2.1 (Presheaf, take I)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is a contravariant functor $\mathcal{F}$ from τ_X to $\mathscr{C}$, i.e. an object of the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$.

For those who do not like the categorical definition, here is an equivalent definition, which just unravels the definition of a functor (cf. Definition A.1.6).

Definition 2.2.2 (Presheaf, take II)**.** A *presheaf* on a topological space X, with values in $\mathscr{C}$, is the assignment $U \mapsto \mathcal{F}(U)$ of an object $\mathcal{F}(U) \in \mathscr{C}$ for each open subset $U \subset X$, and of a morphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ in $\mathscr{C}$ for each inclusion $V \hookrightarrow U$, such that

(1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for every $U \in \tau_X$, and

(2) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for every chain of inclusions $W \hookrightarrow V \hookrightarrow U$.

Terminology 2.2.3. Elements of $\mathcal{F}(U)$ are often called 'sections of $\mathcal{F}$ over U', or (somewhat more vaguely) 'local sections' when $U \subsetneq X$. Elements of $\mathcal{F}(X)$ are called 'global sections', or just 'sections'. Possible alternative notations for $\mathcal{F}(U)$ are $\Gamma(U,\mathcal{F})$ and $\mathrm{H}^0(U,\mathcal{F})$. The maps ρ_{UV} are often called 'restriction maps' (from U to V, the larger set being U).

Notation 2.2.4. Motivated by Terminology 2.2.3, we shall often write $s|_V$ for the image of a section $s \in \mathcal{F}(U)$ along the restriction map ρ_{UV}.

Important Definition 2.2.1 (Sheaf, take I)**.** A *sheaf* on a topological space X, with values in $\mathscr{C}$, is a presheaf $\mathcal{F}$ such that the following two conditions hold:

(3) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$, and two sections $s, t \in \mathcal{F}(U)$ satisfying $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then $s = t$.

(4) Fix an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$ and a tuple $(s_i)_{i\in I}$ of sections $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all $(i, j) \in I \times I$. Then there exists a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$.

Conditions (3) and (4) generalise the conditions (i) and (ii), respectively, anticipated with the example $\mathcal{F} = C^\infty(-,\mathbb{R})$ in Section 2.1.

Terminology 2.2.5. A presheaf $\mathcal{F}$ is called *separated* if Condition (3) holds. Sometimes this condition is called *locality axiom*. Condition (4), on the other hand, is called the *glueing axiom* (or *glueing condition*).

Remark 2.2.6. Let $\mathcal{F}$ be a sheaf. Then, the section $s \in \mathcal{F}(U)$ in the glueing condition (4) is necessarily unique because $\mathcal{F}$ is separated. In fact, the two sheaf conditions could be replaced by a single condition, identical to (4), but imposing uniqueness of s.

Example 2.2.7 (Trivial sheaf)**.** The presheaf defined by $U \mapsto 0$ for every U is a sheaf and is called the *trivial sheaf* (or sometimes the *zero sheaf*). It is simply denoted by '0'.

Example 2.2.8 (Restriction to an open)**.** Let $U \subset X$ be an open subset, $\mathcal{F}$ a presheaf on X. Then, setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for V an open subset of U, defines a presheaf $\mathcal{F}|_U$ on U, which is a sheaf as soon as $\mathcal{F}$ is. It is called the *restriction of $\mathcal{F}$ to U*.

Definition 2.2.9 (Morphism of (pre)sheaves)**.** Let X be a topological space. A *morphism* between two presheaves $\mathcal{F}, \mathcal{G}$ on X is a natural transformation $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$, i.e. a morphism in the functor category $\mathsf{Fun}(\tau_X^{\mathrm{op}}, \mathscr{C})$. A morphism of sheaves is just a morphism between the underlying presheaves.

Let us unravel the definition of natural transformation (cf. Definition A.1.9), to translate Definition 2.2.9 in more concrete terms.

To give a morphism of (pre)sheaves, one has to assign a homomorphism

$$\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

in $\mathscr{C}$ to each $U \in \tau_X$, such that for every inclusion $V \hookrightarrow U$ of open subsets of X, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\eta_U} & \mathcal{G}(U) \\ \rho^{\mathcal{F}}_{UV} \downarrow & & \downarrow \rho^{\mathcal{G}}_{UV} \\ \mathcal{F}(V) & \xrightarrow{\eta_V} & \mathcal{G}(V) \end{array} \tag{2.2.1}$$

commutes. For the sake of clarity, we have emphasised the relevant (pre)sheaf in the restriction maps notation, but we will not be doing that systematically.

Notation 2.2.10. It is clear that presheaves on X with values in $\mathscr{C}$ form a category $\mathsf{pSh}(X,\mathscr{C})$, tautologically defined as

$$\mathsf{pSh}(X,\mathscr{C}) = \mathsf{Fun}(\tau_X^{\mathrm{op}},\mathscr{C}).$$

By Definition 2.2.9, sheaves form a full subcategory, denoted $\mathsf{Sh}(X,\mathscr{C})$. We denote by

$$j_{X,\mathscr{C}} : \mathsf{Sh}(X,\mathscr{C}) \hookrightarrow \mathsf{pSh}(X,\mathscr{C}) \tag{2.2.2}$$

the (fully faithful) inclusion functor.

An isomorphism of (pre)sheaves is an isomorphism in $\mathsf{pSh}(X,\mathscr{C})$, i.e. a *natural isomorphism*, i.e. a natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ such that η_U is an isomorphism in $\mathscr{C}$ for every $U \in \tau_X$ (cf. Definition A.1.10).

Notation 2.2.11. Since (pre)sheaves form a genuine category, from now on we shall use the classical arrow notation '$\mathcal{F} \to \mathcal{G}$' (instead of $\mathcal{F} \Rightarrow \mathcal{G}$) to denote a morphism of (pre)sheaves.

The following definition makes sense, because $\mathscr{C}$ is assumed to be a concrete category.

Definition 2.2.12 (Injective map of presheaves)**.** A morphism of (pre)sheaves $\eta : \mathcal{F} \to \mathcal{G}$ is *injective* if η_U is injective for every U. We denote this by writing η as '$\mathcal{F} \hookrightarrow \mathcal{G}$' (or somewhat more informally '$\mathcal{F} \subset \mathcal{G}$'), and we say that $\mathcal{F}$ is a *sub(pre)sheaf* of $\mathcal{G}$.

We close this section with a few examples and exercises.

Example 2.2.13 (Smooth functions)**.** Let X be a smooth manifold. Then, sending $U \subset X$ to the set $C^\infty(U,\mathbb{R})$ of smooth functions $U \to \mathbb{R}$, defines a sheaf $C^\infty(-,\mathbb{R})$ with values in the category of $\mathbb{R}$-algebras.

Example 2.2.14 (Holomorphic functions)**.** Let X be a complex manifold. Then, sending an open subset $U \subset X$ to the set $\mathscr{O}_X^{\mathrm{h}}(U)$ of holomorphic functions on U, defines a sheaf $\mathscr{O}_X^{\mathrm{h}}$ with values in the category of $\mathbb{C}$-algebras. Sending U to the set $\mathscr{O}_X^{\mathrm{h},\times}(U)$ of nowhere zero holomorphic functions on U defines a sheaf of abelian groups on X (the group structure being given by pointwise multiplication of functions).

Example 2.2.15 (Continuous functions are a sheaf)**.** Let X, Y be topological spaces. For $U \subset X$ open, define

$$\mathcal{F}(U) = \{\, \text{continuous functions } U \to Y \,\}.$$

Then $\mathcal{F}$ is a sheaf of sets on X.

Example 2.2.16 (Separated presheaf, not a sheaf, take I)**.** Set $X = \mathbb{C}$. Then, sending $U \subset X$ to the subset

$$\mathcal{F}(U) = \left\{\, f \in \mathscr{O}_X^{\mathrm{h}}(U) \,\middle|\, f = g^2 \text{ for some } g \in \mathscr{O}_X^{\mathrm{h}}(U) \,\right\}$$

defines a (separated) presheaf. However, $\mathcal{F}$ is not a sheaf: the function $f(z) = z$ on the annulus

$$U = \{\, z \in \mathbb{C} \mid 1 - \varepsilon < |z| < 1 + \varepsilon \,\} \subset \mathbb{C}$$

has a square root in any neighbourhood of any point $x \in U$, but there is no global $g(z) = \sqrt{z}$ defined on the whole of U.

Exercise 2.2.17 (Separated presheaf, not a sheaf, take II)**.** Let $X = \mathbb{R}$, with the standard topology. Show that

$$U \mapsto \mathrm{B}(U) = \{\, \text{bounded continuous functions } U \to \mathbb{R} \,\}$$

is a separated presheaf on X, but not a sheaf (i.e. Condition (4) fails).

Example 2.2.18 (Constant presheaf)**.** Work with $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$, the category of abelian groups, and fix $G \neq 0$ in this category. Fix a topological space X, and define

$$\underline{G}_X^{\mathrm{pre}}(U) = \begin{cases} G & \text{if } U \neq \emptyset, \\ 0 & \text{if } U = \emptyset. \end{cases}$$

As for the restriction maps, set $\rho_{UV} = \mathrm{id}_G$ if both U and V are nonempty. This is a presheaf, which happens to be a sheaf only in precise circumstances (cf. Exercise 2.2.20). For instance, suppose $X = U_1 \amalg U_2$ is a disjoint union of two nonempty open subsets. Then $\underline{G}_X^{\mathrm{pre}}(X) = G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1,2$. Now, $X = U_1 \amalg U_2$ is an open cover. Pick two *distinct* sections $s_i \in G = \underline{G}_X^{\mathrm{pre}}(U_i)$ for $i = 1,2$. Then, $s_1|_{U_1 \cap U_2} = s_1|_{\emptyset} = 0 = s_2|_{\emptyset} = s_2|_{U_1 \cap U_2}$, but there is no section $s \in \underline{G}_X^{\mathrm{pre}}(X) = G$ such that $s|_{U_i} = s_i$ since $\rho_{XU_i} = \mathrm{id}_G$ for $i = 1,2$ and $s_1 \neq s_2$ by assumption. Hence Condition (4), i.e. the gluing axiom, fails (whereas Condition (3) is trivially satisfied). We will see in Example 2.5.3 that $\underline{G}_X^{\mathrm{pre}}$ can be "transformed" into a sheaf by a canonical procedure.

Exercise 2.2.19. Provide examples of presheaves which satisfy the glueing axiom but not the separation axiom.

Exercise 2.2.20. Show that the constant presheaf $\underline{G}_X^{\mathrm{pre}}$ of Example 2.2.18 is a sheaf if and only if every nonempty open subset $U \subset X$ is connected.

Exercise 2.2.21 (Preheaves kernel and cokernel)**.** Let $\mathscr{C}$ be an abelian category, so that every arrow has a kernel and a cokernel. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves with values in $\mathscr{C}$. Consider the assignments

$$U \mapsto (\ker_{\mathrm{pre}} \eta)(U) = \ker(\eta_U)$$
$$U \mapsto (\operatorname{coker}_{\mathrm{pre}} \eta)(U) = \operatorname{coker}(\eta_U) = \mathcal{G}(U)/\operatorname{im}(\eta_U).$$

Show that

(i) both $\ker_{\mathrm{pre}} \eta$ and $\operatorname{coker}_{\mathrm{pre}} \eta$ are presheaves,

(ii) There is a morphism of presheaves $\ker_{\mathrm{pre}} \eta \to \mathcal{F}$ (resp. $\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta$) which satisfies the universal property of the kernel (resp. the cokernel) in $\mathsf{pSh}(X, \mathscr{C})$,

(iii) $\ker_{\mathrm{pre}} \eta$ is a sheaf, denoted $\ker(\eta)$, as soon as η is a morphism of *sheaves*,

(iv) if η is a morphism of sheaves, then $\ker(\eta)$ satisfies the universal property of the kernel in $\mathsf{Sh}(X, \mathscr{C})$, and η is injective if and only if $\ker(\eta) = 0$.

Example 2.2.22 ($\operatorname{coker}_{\mathrm{pre}} \eta$ may not be a sheaf)**.** Let $X = \mathbb{C}$ and $\mathscr{C} = \mathsf{Ab}$. Consider the morphism of sheaves

$$\exp\colon \mathscr{O}_X^{\mathrm{h}} \to \mathscr{O}_X^{\mathrm{h},\times}, \quad f \mapsto \exp\big(f\big),$$

where $\mathscr{O}_X^{\mathrm{h},\times}$ is the sheaf of nowhere zero holomorphic functions (cf. Example 2.2.14). We have that the open subset $U = X \setminus \{0\} \subset X$ is covered by the two open subsets

$$U_1 = X \setminus [0, +\infty] \subset X, \quad U_2 = X \setminus (-\infty, 0] \subset X.$$

The function $g(z) = z$ viewed in $\mathscr{O}_X^{\mathrm{h},\times}(U)$ is not of the form $\exp\big(f\big)$ for any $f \in \mathscr{O}_X^{\mathrm{h}}(U)$. Thus the $\widetilde{g}$ image of g along

$$\mathscr{O}_X^{\mathrm{h},\times}(U) \to \operatorname{coker}_{\mathrm{pre}}(\exp)(U)$$

is nonzero. However, U_1 and U_2 are simply connected, thus every function $h_i \in \mathscr{O}_X^{\mathrm{h},\times}(U_i)$ is of the form $\exp\big(f_i\big)$ for some $f_i \in \mathscr{O}_X^{\mathrm{h}}(U_i)$. Thus $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i) = 0$ for $i = 1, 2$. In particular, the restrictions $g|_{U_i}$ have this property, namely they go (necessarily) to 0 in $\operatorname{coker}_{\mathrm{pre}}(\exp)(U_i)$. If $\operatorname{coker}_{\mathrm{pre}}(\exp)$ were a sheaf, the gluing axiom would force $\widetilde{g} = 0$, which is not true.

2.3 The sheaf condition via equalisers

We now present an alternative way to define sheaves. We will repeatedly use this reinterpretation throughout these notes.

Let $\mathscr{C}$ be a category with limits (cf. Definition B.3.1). In particular, $\mathscr{C}$ has products, equalisers, and a final object (cf. Appendix B.3.1 for full details). The reader may imagine $\mathscr{C}$ to be, for instance, any of the following categories: sets, groups, rings, algebras over a fixed ring, modules over a fixed ring.

Fix a presheaf $\mathcal{F}$ with values in $\mathscr{C}$ on a topological space X. Let $\{U_i\}_{i\in I}$ be a family of open subsets of X, and set $U = \bigcup_{i\in I} U_i$. Then, by our assumption on $\mathscr{C}$, one can consider the map

$$\rho\colon \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_{i\in I},$$

as well as the family of maps

$$\mu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_i|_{U_i\cap U_j}$$

$$\nu_{ij}\colon \prod_{i\in I} \mathcal{F}(U_i) \to \mathcal{F}(U_j) \to \mathcal{F}(U_i \cap U_j), \qquad (s_i)_{i\in I} \mapsto s_j|_{U_i\cap U_j}$$

which, taking products over $(i,j)\in I\times I$, can be assembled into two maps

$$\prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j).$$

Definition 2.3.1 (Sheaf, take II)**.** Let $\mathscr{C}$ be a category with limits, X a topological space. A presheaf $\mathcal{F} \in \mathsf{pSh}(X,\mathscr{C})$ is a *sheaf* if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i\cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Informally, being an equaliser means that ρ is injective and its image agrees with the set of tuples $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ for all pairs (i,j).

Note that Definition 2.3.1 is *element-free*. However, let us check that it agrees with Important Definition 2.2.1 when $\mathscr{C}$ is concrete: in this case the injectivity of ρ, implied by the equaliser condition, coincides with separatedness; the fact that the set-theoretic image of ρ coincides with the collection of tuples of sections $(s_i)_{i\in I}$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ is precisely the glueing condition.

Remark 2.3.2. Let $\mathcal{F}$ be a sheaf. Then, one has $\mathcal{F}(\emptyset) = 0$, the final object in $\mathscr{C}$. This is sometimes listed as an axiom defining a (pre)sheaf, but it does in fact follow from our assumptions (cf. Example B.3.3).

Example 2.3.3. Let $\mathcal{F}$ be a sheaf on X. If $U = \coprod_{i\in I} U_i$ is a *disjoint* union of open subsets $U_i \subset U$, then ρ is an isomorphism, i.e. $s \mapsto (s|_{U_i})_{i\in I}$ defines an isomorphism

$$\rho : \mathcal{F}(U) \xrightarrow{\sim} \prod_{i\in I} \mathcal{F}(U_i).$$

Example 2.3.4. Let $\mathscr{C}$ be an abelian category. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf if for every family of open subsets $\{U_i\}_{i\in I}$, with $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\mu-\nu} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact, where the map denoted $\mu - \nu$ sends $(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$.

The following lemma applies, for instance, to categories of groups, rings, algebras over a ring, and modules over a ring. It allows one to check the sheaf conditions in the category of sets.

LEMMA 2.3.5 ([15, Tag 0073]). *Let $\mathscr{C}$ be a category, $F : \mathscr{C} \to \mathsf{Sets}$ a faithful functor such that $\mathscr{C}$ has limits and F commutes with them. Assume that F reflects isomorphisms. Then a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ is a sheaf of and only if the underlying presheaf of sets $F \circ \mathcal{F} : \tau_X^{\mathrm{op}} \to \mathsf{Sets}$ is a sheaf.*

At the beginning of this chapter we have defined (pre)sheaves of objects in an arbitrary concrete category $\mathscr{C}$. We still have to define a few things, though, e.g. stalks and sheafification. In order for everything to be well-defined and work well (but still be compatible with all we have discussed so far, including Definition 2.3.1), we need to add a few initial data. This is provided by the following definition.

Definition 2.3.6 ([15, Tag 007L])**.** A *type of algebraic structure* is a pair $(\mathscr{C}, F)$, where $\mathscr{C}$ is a category, $F : \mathscr{C} \to \mathsf{Sets}$ is a faithful functor, such that

1. $\mathscr{C}$ has limits and F commutes with them,
2. $\mathscr{C}$ has filtered colimits and F commutes with them,
3. F reflects isomorphisms (i.e. F is *conservative*).

A few remarks are in order, before we go on.

- Equipping a category $\mathscr{C}$ with a faithful functor $F : \mathscr{C} \to \mathsf{Sets}$ is like saying that $\mathscr{C}$ is a *concrete category*, which we had already assumed in Section 2.2.
- If we have a type of algebraic structure $(\mathscr{C}, F)$, then we can verify whether a presheaf is a sheaf in the category of sets, by Lemma 2.3.5.

- The condition that F be conservative implies that a bijective morphism in $\mathscr{C}$ is an isomorphism.
- For every type of algebraic structure $(\mathscr{C}, F)$, one has the following properties:
  (i) $\mathscr{C}$ has a final object 0, and $F(0)$ is a final object in Sets (i.e. a singleton).
  (ii) $\mathscr{C}$ has products, fibre products, and equalisers — this follows from the examples in Appendix B.3.1. Moreover, F commutes with all of them.
- Examples of categories $\mathscr{C}$ having the additional structure of Definition 2.3.6 are:
  - monoids,
  - groups,
  - abelian groups,
  - rings,
  - modules over a ring.

  In all these cases, we take as the functor F the obvious forgetful functor. The reader is encouraged to just think of $\mathscr{C}$ as one of these familiar categories, and not bother too much about Definition 2.3.6. As a counterexample, however, consider the category Top of topological spaces: the forgetful functor to Sets exists but does not reflect isomorphisms (a continuous bijection need not be a homeomorphism).

2.4 Stalks, and what they tell us

Fix a type of algebraic structure $(\mathscr{C}, F\colon \mathscr{C} \to \mathsf{Sets})$ as in Definition 2.3.6. Let X be a topological space, $x \in X$ a point. The collection of open subsets $U \subset X$ containing x forms a directed system (the partial order $\succeq$ being the inclusion relation, i.e. $V \succeq U$ if and only if $V \subset U$). Indeed, given two open neighbourhoods U and V of x, there is always a third open neighbourhood of x contained in both U and V, namely $U \cap V$ or any smaller open subset containing x. In fancier language, the subcategory

$$\iota_x\colon \mathsf{Ngb}_x = \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \hookrightarrow \tau_X^{\mathrm{op}}$$

is a filtered category (see Definition B.3.9).

Important Definition 2.4.1 (Stalks)**.** Let $x \in X$ be a point, $\mathcal{F}$ a presheaf. The *stalk of* $\mathcal{F}$ *at* x is the filtered colimit

$$\mathcal{F}_x = \varinjlim_{\mathsf{Ngb}_x} \mathcal{F} \circ \iota_x = \varinjlim_{U \ni x} \mathcal{F}(U) \in \mathscr{C}.$$

Because F commutes with colimits, the underlying *set* $F(\mathcal{F}_x) \in \mathsf{Sets}$, still denoted $\mathcal{F}_x$, is

$$\mathcal{F}_x = \{(U, s) \mid x \in U,\ s \in \mathcal{F}(U)\}/\sim$$

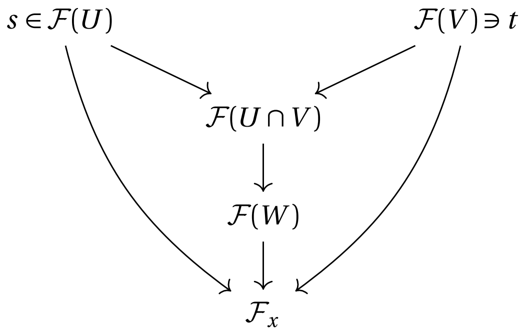
where $(U, s) \sim (V, t)$ whenever there is an open neighbourhood $W \subset U \cap V$ of x such that $s|_W = t|_W$. We denote by

$$s_x = [U, s] \in \mathcal{F}_x$$

the equivalence class of the pair (U, s). It is called the *germ of* s *at* x. By definition of direct limit, there are natural homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad s \mapsto s_x,$$

in $\mathscr{C}$, for every open neighbourhood U of x. The diagram



illustrates the fact that two sections $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ define the same element in the stalk $\mathcal{F}_x$ if and only if there is an intermediate open subset $W \subset U \cap V$ over which they agree.

Figure 2.1: A bunch of sheaves sitting in their natural habitat. The little tops of each leaf of corn are the stalks.

LEMMA 2.4.1. *If $\mathcal{F}$ is a separated presheaf of sets (e.g. a sheaf), then the natural map*

$$\sigma_U^{\mathcal{F}}\colon \mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x, \quad s \mapsto (s_x)_{x\in U} \tag{2.4.1}$$

is injective for every open subset U of X.

The lemma means, at an informal level, that sections are determined by their germs.

Proof. If s and t are sections in $\mathcal{F}(U)$ such that $s_x = t_x$ in $\mathcal{F}_x$ for every $x \in U$, then for every $x \in U$ there is an open neighbourhood $U_x \subset U$ such that $s|_{U_x} = t|_{U_x}$. But this holds for every $x \in U$, and $U = \bigcup_{x\in U} U_x$ is an open covering, thus by the separation axiom we deduce $s = t$, i.e. $\sigma_U^{\mathcal{F}}$ is injective. □

Consider the following property of a tuple $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$, for $U \subset X$ an open subset:

$$\begin{gathered}\text{for every } x\in U \text{ there exists a pair } (V_x, t^x),\\ \text{with } x\in V_x\subset U \text{ and } t^x \in \mathcal{F}(V_x),\\ \text{such that } t^x_y = s_y \text{ for all } y\in V_x.\end{gathered} \tag{2.4.2}$$

Definition 2.4.2 (Compatible germs)**.** Let $\mathcal{F}$ be a presheaf on X, and let $U \subset X$ be an open subset. We say that $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$ is a *tuple of compatible germs* if Condition (2.4.2) is fulfilled.

We always have inclusions

$$\operatorname{im}(\sigma_U^{\mathcal{F}}) \subset \{\text{ tuples } (s_x)_{x\in U} \text{ of compatible germs }\} \subset \prod_{x\in U} \mathcal{F}_x \tag{2.4.3}$$

where the first inclusion is justified by taking $V_x = U$ and $t^x = s$ for every $x \in U$ as soon as $\sigma_U^{\mathcal{F}}(s) = (s_x)_{x\in U}$. If $\mathcal{F}$ is a sheaf, then tuples of compatible germs form precisely the image of the map (2.4.1), i.e. the first inclusion in (2.4.3) is an equality. Indeed, assume $(s_x)_{x\in U}$ consists of compatible germs. Let $\{(V_x, t^x) \mid x \in U\}$ be as in the displayed condition (2.4.2). By the compatibility condition, for every pair $(x, x') \in U \times U$ we have

$$t^x_y = t^{x'}_y, \quad y \in V_x \cap V_{x'}.$$

It follows from Lemma 2.4.1 that

$$t^x\big|_{V_x\cap V_{x'}} = t^{x'}\big|_{V_x\cap V_{x'}}. \tag{2.4.4}$$

Now, we have an open cover $U = \bigcup_{x\in U} V_x$, so by the glueing axiom, applicable by (2.4.4), the sections $t^x \in \mathcal{F}(V_x)$ glue to a (unique) section $t \in \mathcal{F}(U)$ such that $t|_{V_x} = t^x$. But $t^x_y = s_y$ for $y \in V_x$, and this holds for every $x \in U$, so $\sigma_U^{\mathcal{F}}(t) = (s_x)_{x\in U}$.

Summing up, when $\mathcal{F}$ is a sheaf, we have a bijection

$$\sigma_U^{\mathcal{F}} \colon \mathcal{F}(U) \xrightarrow{\sim} \{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \} .$$

This also shows that *sections of a sheaf can always be identified with 'nicely gluable' functions!* Indeed, tuples $(s_x)_{x\in U}$ correspond to particular functions $U \to \coprod_{x\in U} \mathcal{F}_x$, sending $x \in U$ inside the corresponding stalk, and doing so in a compatible way.

LEMMA 2.4.3. *Let $s, t \in \mathcal{F}(X)$ be two global sections of a sheaf $\mathcal{F}$, such that $s_x = t_x \in \mathcal{F}_x$ for every $x \in X$. Then $s = t$.*

Proof. This is just a special case of Lemma 2.4.1. □

Exercise 2.4.4. Let $\mathcal{F}$ be a sheaf on X, and let $s, t \in \mathcal{F}(X)$ be two global sections. Show that

$$\{ x \in X \mid s_x = t_x \} \subset X$$

is an open subset of X.

A morphism of presheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ induces a morphism $\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ at the level of stalks for every $x \in X$, defined by

$$s_x = [U, s] \mapsto [U, \eta_U(s)] = (\eta_U(s))_x . \tag{2.4.5}$$

Exercise 2.4.5. Check that (2.4.5) is well-defined.

If $U \subset X$ is an open subset containing a point $x \in X$, then the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\eta_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\eta_x} & \mathcal{G}_x \end{array} \qquad \begin{array}{ccc} s & \overset{\eta_U}{\longmapsto} & \eta_U(s) \\ \downarrow & & \downarrow \\ s_x & \overset{\eta_x}{\longmapsto} & (\eta_U(s))_x \end{array}$$

commutes. What we have just said can be rephrased by saying that the association $\mathcal{F} \mapsto \mathcal{F}_x$ defines a functor

$$\mathsf{stalk}_x \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathscr{C} . \tag{2.4.6}$$

We will see that in reasonable circumstances the restriction of this functor to the category of sheaves is *exact* (cf. Proposition 2.5.14).

Definition 2.4.6. A morphism of (pre)sheaves $\eta \colon \mathcal{F} \to \mathcal{G}$ is *surjective* if η_x is surjective for every $x \in X$.

Warning 2.4.7. You may have noticed that surjectivity of a map of sheaves (cf. Definition 2.4.6) is defined differently than injectivity (cf. Definition 2.2.12)!

Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then

$$\eta_x \text{ is surjective} \iff \begin{array}{c}\text{for every } t_x \in \mathcal{G}_x \text{ there exists an open neighbourhood} \\ U \text{ of } x \text{ and a section } s \in \mathcal{F}(U) \text{ such that } (\eta_U(s))_x = t_x.\end{array}$$

$$\eta \text{ is surjective} \iff \begin{array}{c}\text{for every open subset } U \subset X \text{ and for every} \\ t \in \mathcal{G}(U), \text{ there exists a covering } U = \bigcup_{i\in I} U_i \\ \text{such that } t|_{U_i} \text{ is in the image of } \eta_{U_i} \text{ for every } i.\end{array}$$

The second equivalence is obtained as follows.

Proof of '⇒'. Assume η is surjective, i.e. η_x is surjective for every $x \in X$. Fix $U \subset X$ open and a local section $t \in \mathcal{G}(U)$. For every $x \in U$, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{\eta_U} & \mathcal{G}(U) & \qquad & t \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\eta_x} & \mathcal{G}_x & & t_x \end{array}$$

where $t_x \in \mathcal{G}_x$ can be lifted along η_x to an element $s_x \in \mathcal{F}_x$. Let (V_x, s) be a representative for s_x, so that in particular $s \in \mathcal{F}(V_x)$. The identity $\eta_x(s_x) = t_x$ implies that there is an open neighbourhood $x \in U_x \subset V_x \cap U$ such that

$$\eta_{U_x}(s|_{U_x}) = t|_{U_x}.$$

Now this holds for every $x \in U$, and the elements of $\{\, U_x \mid x \in U \,\}$ form a covering of U, thus we have proved the condition.

Proof of '⇐'. Conversely, assuming the condition, let us prove surjectivity of η. Fix $x \in X$ along with a germ $t_x \in \mathcal{G}_x$. We need to prove that t_x has a preimage in $\mathcal{F}_x$. Let (U, t) be a representative of t_x, so that $t \in \mathcal{G}(U)$. By the condition we are assuming, there exists a covering $U = \bigcup_{i\in I} U_i$ such that $t|_{U_i} = \eta_{U_i}(s_i)$ for some $s_i \in \mathcal{F}(U_i)$, for every $i \in I$. If $x \in U_i$, we have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{F}(U_i) & \xrightarrow{\eta_{U_i}} & \mathcal{G}(U_i) & \qquad & s_i & \longmapsto & t|_{U_i} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\eta_x} & \mathcal{G}_x & & \star & \longmapsto & t_x \end{array}$$

so the element $\star \in \mathcal{F}_x$ is a preimage of t_x. The equivalence is proved.

The next result incarnates the local nature of sheaves.

LEMMA 2.4.8. *Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The following are equivalent:*

(i) *η is an isomorphism,*

(ii) *η_x is an isomorphism for every $x \in X$,*

(iii) *η is injective and surjective.*

Proof. Recall that η is an isomorphism if and only if η_U is an isomorphism for every U.

Proof of (i) $\Rightarrow$ (ii). By functoriality of $\mathcal{F} \mapsto \mathcal{F}_x$, we have that if η is an isomorphism, then so is η_x for every $x \in X$.

Proof of (ii) $\Rightarrow$ (i). Suppose η_x is an isomorphism for every x. Let $U \subset X$ be an open subset: we need to show that η_U is an isomorphism.

To see that η_U is injective, pick $s, t \in \mathcal{F}(U)$ such that $\eta_U(s) = \eta_U(t) \in \mathcal{G}(U)$. Then, for any $x \in U$, one has

$$\eta_x(s_x) = (\eta_U(s))_x = (\eta_U(t))_x = \eta_x(t_x),$$

which implies $s_x = t_x$ by injectivity of η_x. This holds for every $x \in U$ by assumption, thus $s = t$ by Lemma 2.4.3. Therefore, η_U is injective for every U (i.e. η is injective).

To see that η_U is surjective, pick $t \in \mathcal{G}(U)$. By surjectivity of η (which we have by definition since η_x is surjective for every $x \in X$), we can find an open cover $U = \bigcup_{i\in I} U_i$ along with a collection of sections $s_i \in \mathcal{F}(U_i)$ such that $\eta_{U_i}(s_i) = t|_{U_i}$. But by the previous paragraph η is injective, so s_i and s_j agree on $U_i \cap U_j$. Therefore, since $\mathcal{F}$ is a sheaf, they glue to a section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$. By construction, $\eta_U(s)|_{U_i} = \eta_{U_i}(s_i) = t|_{U_i}$, which implies $\eta_U(s) = t$ since $\mathcal{G}$ is a sheaf. Thus η_U is surjective.

Proof of (ii) $\Rightarrow$ (iii). The first paragraph of '(ii) $\Rightarrow$ (i)' already shows that if η_x is an isomorphism for every $x \in X$, then η_U is injective for all U, i.e. η is injective. Surjectivity follows from the definition.

Proof of (iii) $\Rightarrow$ (ii). We only need to show that if η_U is injective for every U, then η_x is injective for every $x \in X$. Consider $s_x = [U, s]$ and $s'_x = [U', s']$ two germs in $\mathcal{F}_x$ such that $\eta_x(s_x) = \eta_x(s'_x)$ in $\mathcal{G}_x$. Then there is an open subset $W \subset U \cap U'$ such that $\eta_U(s)|_W = \eta_{U'}(s')|_W$. But by compatibility of η_W with restrictions, this is equivalent to the identity $\eta_W(s|_W) = \eta_W(s'|_W)$, which by our assumption implies $s|_W = s'|_W$. But then $s_x = s'_x$. □

Warning 2.4.9. It is not true that two sheaves with isomorphic stalks are isomorphic: there may be no map between them! For instance, consider a topological space X consisting of two points x_0, x_1 where only x_0 is a closed point. Thus X and $U = X \setminus \{x_0\}$ are the only nonempty open subsets of X. Fix an abelian group $G \neq 0$ and define $\mathcal{F}(X) = G = \mathcal{F}(U)$. Then choose either $\rho_{XU} = \mathrm{id}_G$ or $\rho_{XU} = 0$ to define two distinct sheaves on X. They have the same stalks but they are not isomorphic.

Exercise 2.4.10. Show that Lemma 2.4.8 fails for presheaves.

Example 2.4.11 (Surjectivity is subtle)**.** Let $\mathcal{F} = \mathscr{O}_X^{\mathrm{h}}$ be the sheaf of holomorphic functions on $X = \mathbb{C} \setminus \{0\}$, and let $\mathcal{G} = \mathcal{F}^\times$ be the sheaf of invertible holomorphic functions on X. The map $\exp\colon \mathcal{F} \to \mathcal{G}$ is surjective, but $\exp_X\colon \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective, e.g. the function $f(z) = z$ in $\mathcal{G}(X)$ is not the exponential of a homolomorphic function (cf. Example 2.2.22).

Example 2.4.12 (Skyscraper sheaf)**.** Let X be a topological space, G a nontrivial abelian group, $x \in X$ a point. The assignment

$$U \mapsto G_x(U) = \begin{cases} G & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

defines a sheaf of abelian groups, choosing as restriction maps the identity of G or the zero map in the obvious way. This sheaf is called the *skyscraper sheaf* attached to (X, x, G). At the level of stalks, one has

$$(G_x)_y = \begin{cases} G & \text{if } y \in \overline{\{x\}} \\ 0 & \text{if } y \notin \overline{\{x\}}, \end{cases}$$

because if y is in the closure of x then every neighbourhood of y also contains x, whereas if y is *not* in the closure of x, then $U = X \setminus \overline{\{x\}}$ is the largest open neighbourhood of y and thus $(G_x)_y = 0$ since $G_x(U) = 0$. Thus G_x has only one nonzero stalk (at x) if and only if x is a closed point. This is the case where the name 'skyscraper sheaf' for G_x fits best.

Exercise 2.4.13. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a *sheaf*, and let $\eta_1, \eta_2 \colon \mathcal{F} \to \mathcal{G}$ be two morphisms of presheaves of sets such that $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$. Show that $\eta_1 = \eta_2$. Show that it is in fact necessary to assume $\mathcal{G}$ to be a sheaf. This exercise will be needed in Theorem 3.1.61.

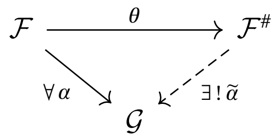
2.5 Sheafification

Fix a type of algebraic structure $(\mathscr{C}, F \colon \mathscr{C} \to \mathsf{Sets})$. Friendly translation: fix $\mathscr{C}$ to be either of the following categories:

- monoids,
- groups,
- abelian groups,
- rings,
- modules over a ring.

Let X be a topological space. Let $\mathcal{F} \colon \tau_X^{\mathrm{op}} \to \mathscr{C}$ be a presheaf. We next define a *sheaf* $\mathcal{F}^\#$, called the sheafification of $\mathcal{F}$, via an explicit universal property, and having precisely the same stalks as the initial presheaf $\mathcal{F}$.

Definition 2.5.1 (Sheafification of a presheaf)**.** Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. A *sheafification* of $\mathcal{F}$ is a pair $(\mathcal{F}^\#, \theta)$, where $\mathcal{F}^\# \in \mathsf{Sh}(X, \mathscr{C})$ is a sheaf and $\theta \colon \mathcal{F} \to \mathcal{F}^\#$ is a morphism of presheaves, such that for every other pair $(\mathcal{G}, \alpha)$ where $\mathcal{G}$ is a sheaf and $\alpha \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there exists a unique morphism of sheaves $\widetilde{\alpha} \colon \mathcal{F}^\# \to \mathcal{G}$ such that $\alpha = \widetilde{\alpha} \circ \theta$.



PROPOSITION 2.5.2. *Let $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ be a presheaf. Then a sheafification $(\mathcal{F}^\#, \theta)$ exists, and the map $\theta_x \colon \mathcal{F}_x \to \mathcal{F}^\#_x$ is an isomorphism for every $x \in X$.*

What follows immediately from Proposition 2.5.2 is that $\mathcal{F}^\#$ is unique up to a unique isomorphism, and moreover the canonical map $\theta \colon \mathcal{F} \to \mathcal{F}^\#$ is an isomorphism precisely when $\mathcal{F}$ is already a sheaf.

Proof. Let $U \subset X$ be an open subset. Define

$$\mathcal{F}^\#(U) = \left\{ \text{functions } U \xrightarrow{f} \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{c} \text{for every } x \in U,\ f(x) \in \mathcal{F}_x \text{ and there exist an} \\ \text{open neighbourhood } V \subset U \text{ of } x \text{ and } s \in \mathcal{F}(V) \\ \text{such that } f(y) = s_y \text{ for every } y \in V \end{array} \right\}.$$

Note that, since $\mathscr{C}$ has products, we can view a function f as above as a tuple

$$(f(x))_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$$

and we can rephrase the definition of $\mathcal{F}^\#(U)$ by saying that

$$\mathcal{F}^\#(U) = \{ \text{tuples } (s_x)_{x \in U} \text{ of compatible germs} \}.$$

See Definition 2.4.2 for the definition of compatible germs. Functoriality of the assignment $U \mapsto \mathcal{F}^\#(U)$ is clear (functions restrict!), thus $\mathcal{F}^\#$ is a presheaf. The morphism $\theta_U \colon \mathcal{F}(U) \to \mathcal{F}^\#(U)$ defined by sending $s \in \mathcal{F}(U)$ to the function

$$f_s \colon U \to \coprod_{x \in U} \mathcal{F}_x, \quad x \mapsto s_x = [U, s] \in \mathcal{F}_x$$

determines a morphism of presheaves, being compatible with restrictions. It is just the function $\sigma_U^{\mathcal{F}}$ introduced in (2.4.1)!

The presheaf $\mathcal{F}^\#$ is a sheaf: Fix an open cover $U = \bigcup_{i \in I} U_i$ of some open subset $U \subset X$ and a collection of sections $f_i \in \mathcal{F}^\#(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i and j. We need to find a unique $f \in \mathcal{F}^\#(U)$ such that $f|_{U_i} = f_i$. Define

$$f \in \prod_{x \in U} \mathcal{F}_x = \operatorname{Hom}\left(U, \coprod_{x \in U} \mathcal{F}_x\right)$$

by the rule

$$f(x) = f_i(x) \in \mathcal{F}_x, \quad x \in U_i \subset U.$$

This is well-defined since, even though x can lie in more than one open U_i, by assumption we have $f_i(x) = f_j(x)$ as soon as $x \in U_i \cap U_j$. We need to check that f defines an element of the subset $\mathcal{F}^\#(U) \subset \prod_{x\in U} \mathcal{F}_x$. But for every $i \in I$ we know the following: for every $x \in U_i$ there exist an open neighbourhood $x \in V_i \subset U_i$ and a section $s_i \in \mathcal{F}(V_i)$ such that $f(y) = f_i(y) = (s_i)_y$ for all $y \in V_i$. But V_i is also open in U, so the condition defining $\mathcal{F}^\#(U)$ also holds for f. Thus $f \in \mathcal{F}^\#(U)$ satisfies $f|_{U_i} = f_i$, and is clearly unique with this property.

The pair $(\mathcal{F}^\#, \theta)$ *is the sheafification.* Assume we have a sheaf $\mathcal{G}$ and a morphism of presheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$. We need to define a morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$ of presheaves such that $\alpha = \widetilde{\alpha} \circ \theta$. For every U open in X, we need to define a morphism $\widetilde{\alpha}_U : \mathcal{F}^\#(U) \to \mathcal{G}(U)$ in such a way that $\alpha_U = \widetilde{\alpha}_U \circ \theta_U$. Fix $s = (s_x)_{x\in U} \in \mathcal{F}^\#(U)$. The composition

$$\begin{array}{ccc} U & \xrightarrow{\;s\;} & \coprod_{x\in U} \mathcal{F}_x \\ & {\scriptstyle \widetilde{\alpha}_U(s)} \searrow & \Big\downarrow {\scriptstyle \coprod_{x\in U} \alpha_x} \\ & & \coprod_{x\in U} \mathcal{G}_x \end{array}$$

defines a tuple of compatible germs for $\mathcal{G}$ over U, hence an element $\widetilde{\alpha}_U(s) \in \mathcal{G}^\#(U) = \mathcal{G}(U)$, using that $\mathcal{G}$ is a sheaf for this identity. This is the required morphism $\widetilde{\alpha}\colon \mathcal{F}^\# \to \mathcal{G}$.

The map θ *is an isomorphism on stalks.* The map θ, at the level of stalks, is defined by

$$\theta_x[U, s] = [U, f_s].$$

Injectivity: Suppose $\theta_x[U, s] = \theta_x[V, t]$ for two classes $[U, s], [V, t] \in \mathcal{F}_x$, i.e. assume $[U, f_s] = [V, f_t]$ in $\mathcal{F}^\#_x$. Then, by definition of germ, there exists an open neighbourhood $W \subset U \cap V$ of x such that $f_s|_W = f_t|_W$. But this means, by definition of f_s and f_t, that $s_y = t_y$ for all $y \in W$. Thus, in particular, $s_x = t_x$. But this is just the equality $[U, s] = [V, t]$ we were after.

Surjectivity: Pick a class $[U, f] \in \mathcal{F}^\#_x$ for some $f \in \mathcal{F}^\#(U)$ and open neighbourhood U of x. Then, for every $z \in U$, there exist an open neighbourhood $V \subset U$ of z and a section $s \in \mathcal{F}(V)$ such that $f(y) = s_y$ in $\mathcal{F}_y$ for every $y \in V$. We claim that $[U, f] = \theta_x(s_x)$, where $s_x = [V, s]$. Indeed, $\theta_x(s_x) \in \mathcal{F}^\#_x$ is the equivalence class of the map

$$f_s\colon V \to \coprod_{y\in V} \mathcal{F}_y, \quad y \mapsto s_y.$$

But this map agrees with the restriction of f to $V \subset U$ (by the condition $f(y) = s_y$ recalled above), i.e. $f_s = f|_V \in \mathcal{F}^\#(V)$. Since V is also an open neighbourhood of x, it follows that $(f|_V)_x = (f_s)_x = [V, f_s] = \theta_x(s_x) \in \mathcal{F}^\#_x$, but of course $(f|_V)_x = [U, f]$. Thus θ_x is surjective. □

Example 2.5.3 (Constant sheaf)**.** Let G be a nontrivial abelian group. The *constant sheaf* on a topological space X, with values in G, is the sheafification $\underline{G}_X$ of the presheaf $\underline{G}_X^{\mathrm{pre}}$ defined in Example 2.2.18. This sheaf agrees with the sheaf whose sections over U are the locally constant functions $U \to G$. This, in turn, agrees with the following: endow G with the discrete topology and consider the assignment

$$U \mapsto \{\text{continuous maps } U \to G\},$$

which we know is a sheaf by Example 2.2.15. If $U \subset X$ is a connected open subset, then $\underline{G}_X(U) = G$. By Proposition 2.5.2, at the level of stalks we have $\underline{G}_{X,x} = G$ for every $x \in X$, since the stalks of the constant presheaf are manifestly all equal to G.

Exercise 2.5.4. Let X be a connected topological space, x a point, G a nontrivial abelian group. Under what condition(s) is the constant sheaf $\underline{G}_X$ equal to the skyscraper sheaf G_x (cf. Example 2.4.12)?

Exercise 2.5.5. Show that sending $\mathcal{F} \mapsto \mathcal{F}^\#$ defines a functor $(-)^\# \colon \mathsf{pSh}(X, \mathscr{C}) \to \mathsf{Sh}(X, \mathscr{C})$, and that the forgetful functor $j_{X,\mathscr{C}} \colon \mathsf{Sh}(X, \mathscr{C}) \hookrightarrow \mathsf{pSh}(X, \mathscr{C})$ is a right adjoint. This means (cf. Definition A.1.17) that are bifunctorial bijections

$$\psi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(\mathcal{F}^\#, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(\mathcal{F}, \mathcal{G}), \quad \widetilde{\alpha} \mapsto \widetilde{\alpha} \circ \theta$$

for any presheaf $\mathcal{F}$ and sheaf $\mathcal{G}$. (**Hint**: the universal property of the sheafification!).

2.5.1 Subsheaves, Quotient sheaves

We have essentially already proved the following general result.

PROPOSITION 2.5.6 ([15, Tag 007S]). *Let X be a topological space. Let $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X, \mathsf{Sets})$ be sheaves of sets, $\eta \colon \mathcal{F} \to \mathcal{G}$ a morphism. Then, the following are equivalent:*

(a) *η is a monomorphism,*

(b) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$,*

(c) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subset X$ (i.e. η is injective).*

Furthermore, the following are equivalent:

(i) *η is an epimorphism,*

(ii) *$\eta_x \colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$ (i.e. η is surjective),*

and are implied (but not equivalent to, cf. Example 2.4.11!) by the condition

(iii) *$\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open subsets $U \subset X$.*

If $\mathscr{C}$ is an abelian category (e.g. Mod_A for a fixed ring A), then Proposition 2.5.6 holds replacing Sets with $\mathscr{C}$.

Definition 2.5.7 (Subsheaf, quotient sheaf)**.** If there exists a morphism of sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ such that either of the equivalent conditions (a), (b) or (c) holds, we say that $\mathcal{F}$ is a *subsheaf* of $\mathcal{G}$ (and we may denote this by '$\mathcal{F} \subset \mathcal{G}$'). If either of the equivalent conditions (i) or (ii) holds, we say that $\mathcal{G}$ is a *quotient sheaf* of $\mathcal{F}$.

Example 2.5.8 (Quotient by a subsheaf)**.** Let $\mathscr{C}$ be an abelian category. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf (with values in $\mathscr{C}$), then sending

$$U \mapsto \mathcal{G}(U)/\mathcal{F}(U) \tag{2.5.1}$$

is a presheaf on X, because the restriction maps respect the inclusions $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$, and thus pass to the quotients. Its sheafification $\mathcal{G}/\mathcal{F}$ is called the *quotient sheaf of* $\mathcal{G}$ *by* $\mathcal{F}$. There is a natural morphism of sheaves $\mathcal{G} \to \mathcal{G}/\mathcal{F}$.

Definition 2.5.9 (Sheaf image, sheaf cokernel)**.** Let $\mathscr{C}$ be an abelian category, $\eta\colon \mathcal{F} \to \mathcal{G}$ a morphism of sheaves (with values in $\mathscr{C}$), so that $\ker(\eta) \hookrightarrow \mathcal{F}$ is a subsheaf by Exercise 2.2.21. The sheafification $\operatorname{im}(\eta)$ of the presheaf

$$U \mapsto \operatorname{im}_{\mathrm{pre}}(U) = \operatorname{im}(\eta_U) = \mathcal{F}(U)/\ker(\eta_U)$$

is called the *image of* η. It is a special case of Example 2.5.8 and defines a subsheaf

$$\operatorname{im}(\eta) = \mathcal{F}/\ker(\eta) \subset \mathcal{G}.$$

The quotient sheaf

$$\operatorname{coker}(\eta) = \mathcal{G}/\operatorname{im}(\eta),$$

again a special case of Example 2.5.8, is called the *sheaf cokernel*.

Exercise 2.5.10. Let $\mathscr{C}$ be an abelian category. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves with values in $\mathscr{C}$. Show that the composition

$$\mathcal{G} \to \operatorname{coker}_{\mathrm{pre}} \eta \to \operatorname{coker}(\eta),$$

where the first morphism is given by the natural maps $\mathcal{G}(U) \twoheadrightarrow \mathcal{G}(U)/\operatorname{im}(\eta_U)$ and the last morphism is the sheafification, is a cokernel in the category $\mathsf{Sh}(X, \mathscr{C})$.

Remark 2.5.11. Set $\mathscr{C} = \mathsf{Mod}_A$ (or any Grothendieck abelian category so that, by definition, filtered colimits exist and are exact). Let $\mathcal{F} \subset \mathcal{G}$ be a subsheaf, $x \in X$ a point. Then

$$(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x \tag{2.5.2}$$

in Mod_A. This follows from the fact that $(\mathcal{G}/\mathcal{F})_x$ agrees with the stalk of the *presheaf* (2.5.1), and from right exactness of filtered colimits. Moreover, if $\eta\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves and $x \in X$ is a point, then

$$\begin{aligned} \ker(\eta)_x &= \ker(\eta_x) \\ \operatorname{im}(\eta)_x &= \operatorname{im}(\eta_x) \\ \operatorname{coker}(\eta)_x &= \operatorname{coker}(\eta_x). \end{aligned} \tag{2.5.3}$$

The first identity in (2.5.3) follows from the fact that filtered colimits are *also left exact* in Mod_A, thus

$$\begin{aligned} \ker\left(\mathcal{F}_x \xrightarrow{\eta_x} \mathcal{G}_x\right) &= \ker\left(\varinjlim_{U \ni x} \mathcal{F}(U) \to \varinjlim_{U \ni x} \mathcal{G}(U)\right) \\ &= \varinjlim_{U \ni x} \ker(\mathcal{F}(U) \to \mathcal{G}(U)) \\ &= \ker(\eta)_x. \end{aligned}$$

The last two identities in (2.5.3) are a special case of (2.5.2).

THEOREM 2.5.12 ([6, §10]). *If $\mathscr{C}$ is a Grothendieck abelian category, then $\mathsf{Sh}(X,\mathscr{C})$ is a Grothendieck abelian category.*

Definition 2.5.13. A *short exact sequence of sheaves* with values in a Grothendieck abelian category $\mathscr{C}$ is a short exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

of objects in the abelian category $\mathsf{Sh}(X,\mathscr{C})$. Explicitly, exactness means that ι is injective, π is surjective and $\operatorname{im}(\iota) = \ker(\pi)$.

PROPOSITION 2.5.14. *Let $\mathscr{C}$ be a Grothendieck abelian category. A sequence*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0$$

of objects in $\mathsf{Sh}(X,\mathscr{C})$ is a short exact sequence if and only if

$$0 \longrightarrow \mathcal{F}_x \xrightarrow{\iota_x} \mathcal{G}_x \xrightarrow{\pi_x} \mathcal{H}_x \longrightarrow 0$$

is a short exact sequence in $\mathscr{C}$ for every $x \in X$.

Proof. Combine Remark 2.5.11 and Lemma 2.4.8 with one another. □

Exercise 2.5.15. Let $\eta\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of A-modules, for A a ring. Prove that there is an exact sequence of sheaves

$$0 \longrightarrow \ker(\eta) \longrightarrow \mathcal{F} \xrightarrow{\eta} \mathcal{G} \longrightarrow \operatorname{coker}(\eta) \longrightarrow 0.$$

In particular, if η is injective, this reduces to

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0.$$

Exercise 2.5.16. Let A be a ring. For a nonempty open subset U of a topological space X, consider the functor $\Gamma(U,-)\colon \mathsf{Sh}(X,\mathsf{Mod}_A) \to \mathsf{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ into an exact sequence of A-modules

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U).$$

When $U = X$, this functor takes $\mathcal{F} \mapsto \mathcal{F}(X)$ and is thus called the *global section functor*. Another notation used for it in the literature is $\mathrm{H}^0(X,-)$, cf. Terminology 2.2.3.

2.6 Supports

Let A be a ring. Let $\mathcal{F} \in \mathsf{Sh}(X,\mathsf{Mod}_A)$ be a sheaf of A-modules on a topological space X. Let $U \subset X$ be an open subset, and fix a section $s \in \mathcal{F}(U)$. We have two notions of support: the support of $\mathcal{F}$, and the support of s, defined respectively as

$$\begin{aligned} \operatorname{Supp}(\mathcal{F}) &= \{\, x \in X \mid \mathcal{F}_x \neq 0 \,\}, \\ \operatorname{Supp}(s) &= \{\, x \in U \mid s_x \neq 0 \text{ in } \mathcal{F}_x \,\}. \end{aligned} \tag{2.6.1}$$

If $s_x = 0$, then there is an open neighbourhood $x \in V \subset U$ such that $s|_V = 0 \in \mathcal{F}(V)$. Thus $V \subset U \setminus \operatorname{Supp}(s)$ and hence $\operatorname{Supp}(s) \subset U$ is closed. In fact, this follows from (or solves) Exercise 2.4.4. In general, however, $\operatorname{Supp}(\mathcal{F}) \subset X$ is *not* closed, as the two next examples show.

Example 2.6.1 (Supp($\mathcal{F}$) need not be closed, take I)**.** Let X be an irreducible topological space. This means that any two nonempty open subsets of X intersect. Fix a nontrivial abelian group G, a point $x \in X$, and for $U \in \tau_X$ define

$$\mathcal{F}(U) = \begin{cases} 0 & \text{if } U = \emptyset \text{ or } x \in U \\ G & \text{otherwise.} \end{cases}$$

Let $\rho_{UV} \in \{\mathrm{id}_G, 0\}$ be chosen in the obvious way for all $U, V \in \tau_X$. Then $\mathcal{F}$ is a sheaf of abelian groups on X, with stalks

$$\mathcal{F}_y = \begin{cases} 0 & \text{if } y \in \overline{\{\, x \,\}} \\ G & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{Supp}(\mathcal{F}) = X \setminus \overline{\{\, x \,\}},$$

which is not closed in X as soon as $\overline{\{\, x \,\}} \hookrightarrow X$ is not open.

Example 2.6.2 (Supp($\mathcal{F}$) need not be closed, take II)**.** Let $j: U \hookrightarrow X$ be the inclusion of an open subset U of a topological space X. Let $\mathcal{F} \in \mathsf{Sh}(U, \mathscr{C})$ be a sheaf. Define $j_!\mathcal{F} \in \mathsf{Sh}(X, \mathscr{C})$ to be the sheafification of the presheaf $j_!^{\text{pre}}\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ defined by

$$j_!^{\text{pre}}\mathcal{F}(V) = \begin{cases} \mathcal{F}(V) & \text{if } V \subset U \\ 0 & \text{otherwise.} \end{cases}$$

One has

$$(j_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise,} \end{cases}$$

so that $\text{Supp}(j_!\mathcal{F}) = \text{Supp}(\mathcal{F})$. Let now $\mathscr{C} = \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}}$ be the category of abelian groups. Fix $G \neq 0$ in $\mathscr{C}$ and consider the constant sheaf on U (cf. Example 2.5.3). We have $\text{Supp}(j_!\underline{G}_U) = \text{Supp}(\underline{G}_U) = U$. In particular, $\text{Supp}(j_!\underline{G}_U) \subset X$ is not closed as soon as U is not closed in X.

If $\mathcal{F}$ is a sheaf of rings, the notions of support defined in (2.6.1) still make sense, and one has $\text{Supp}(\mathcal{F}) = \text{Supp}(1)$, where $1 \in \mathcal{F}(X)$ is the ring identity (recall that the '0 ring' is the one where $1 = 0$). Thus $\text{Supp}(\mathcal{F})$ *is* in fact closed in this case.

2.7 Sheaves = sheaves on a base

Fix a type of algebraic structure $(\mathscr{C}, F: \mathscr{C} \to \mathsf{Sets})$.

Definition 2.7.1 (Base of open sets)**.** Let X be a topological space. A *base of open sets* for X is a collection of open subsets $\mathcal{B} \subset \tau_X$ satisfying the following requirements:

(a) $\mathcal{B}$ is stable under finite intersections,

(b) every $U \in \tau_X$ can be written as a union of open sets belonging to $\mathcal{B}$.

Definition 2.7.2 ($\mathcal{B}$-sheaf)**.** A $\mathcal{B}$-*presheaf* (resp. $\mathcal{B}$-*sheaf*) is an assignment

$$U \mapsto \mathcal{F}(U) \in \mathscr{C}, \text{ for each } U \in \mathcal{B},$$

such that the presheaf conditions (1)–(2) of Definition 2.2.1 (resp. the presheaf conditions (1)–(2) of Definition 2.2.1 and the sheaf conditions (3)–(4) of Important Definition 2.2.1) hold, considering only open sets belonging to $\mathcal{B}$.

Notation 2.7.3. We shall use the notation $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ to denote a $\mathcal{B}$-(pre)sheaf.

Note that restriction maps

$$\rho_{UV}: \mathcal{F}(U) \to \mathcal{F}(V)$$

are part of the data of a $\mathcal{B}$-(pre)sheaf whenever $V \subset U$ is an inclusion of open sets both belonging to $\mathcal{B}$. Note, also, that condition (a) in Definition 2.7.1 ensures that open subsets of the form $U \cap V$ belong to $\mathcal{B}$ for all $U, V \in \mathcal{B}$. In particular, as in Section 2.3, a $\mathcal{B}$-presheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a sheaf precisely when the following condition is fulfilled: for every open subset $U \in \mathcal{B}$ and for any open cover $U = \bigcup_{i\in I} U_i$ with all $U_i \in \mathcal{B}$, the diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \overset{\mu}{\underset{\nu}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is an equaliser diagram in $\mathscr{C}$.

Remark 2.7.4. Let $x \in X$ be a point. The collection of open neighbourhoods

$$\mathcal{B}_x = \{ U \in \mathcal{B} \mid x \in U \}^{\mathrm{op}} \subset \tau_X^{\mathrm{op}}$$

is a fundamental system of open neighbourhoods of x, also called a local basis at x (i.e. for any $W \in \mathsf{Ngb}_x$ there exists $U \in \mathcal{B}_x$ such that $U \subset W$). In more technical terms, one may say that the filtered categories Ngb_x and $\mathcal{B}_x$ are *cofinal*, i.e. the inclusion $\mathcal{B}_x \hookrightarrow \mathsf{Ngb}_x$ is a cofinal functor. We will not use this terminology.

By Remark 2.7.4, the stalk

$$\mathcal{F}_x = \varinjlim_{\mathcal{B}_x} \mathcal{F}\big|_{\mathcal{B}_x} = \varinjlim_{U\in\mathcal{B}_x} \mathcal{F}(U) \in \mathscr{C}$$

of a $\mathcal{B}$-(pre)sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ at a point $x \in X$ is well-defined as an object of $\mathscr{C}$. It receives, by definition of direct limit, canonical morphisms

$$\mathcal{F}(U) \to \mathcal{F}_x, \quad U \in \mathcal{B}_x.$$

We denote by $s_x \in \mathcal{F}_x$, as ever, the image of $s \in \mathcal{F}(U)$ under this morphism.

Moreover, if $U \in \mathcal{B}$ and $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ is a $\mathcal{B}$-sheaf, the natural map

$$\begin{aligned} \mathcal{F}(U) &\xrightarrow{\sigma_U^{\mathcal{F}}} \prod_{x\in U} \mathcal{F}_x \\ s &\longmapsto (s_x)_{x\in U} \end{aligned}$$

is injective (as in Lemma 2.4.1), and its image agrees with the collections of compatible germs; to be more precise, we should now call them '$\mathcal{B}$-compatible', for they are, by definition, those tuples

$$(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{F}_x$$

such that for every $x \in U$ there is a pair (V_x, t^x), where $V_x \in \mathcal{B}_x$ satisfies $V_x \subset U$ and $t^x \in \mathcal{F}(V_x)$ satisfies $t^x_y = s_y$ for every $y \in V_x$.

Definition 2.7.5 (Morphism of $\mathcal{B}$-sheaves)**.** A *morphism* of $\mathcal{B}$-(pre)sheaves

$$\eta_{\mathcal{B}} \colon \left\{ \mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{F}} \right\} \longrightarrow \left\{ \mathcal{G}(\mathcal{B}), \rho_{\mathcal{B}}^{\mathcal{G}} \right\} \tag{2.7.1}$$

is the datum of a collection of maps $\eta_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each $U \in \mathcal{B}$, such that Diagram (2.2.1) commutes for all $U, V \in \mathcal{B}$ such that $V \subset U$.

With this definition, $\mathcal{B}$-sheaves form a category, denoted $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$.[1]

Remark 2.7.6. Let X be a topological space, $\mathcal{B}$ a base of open subsets of X. A (pre)sheaf $\mathcal{F}$ on X is a $\mathcal{B}$-(pre)sheaf in a natural way. More precisely, there is (say, at the level of sheaves) a *restriction functor*

$$\mathsf{res}_{\mathcal{B}}(X, \mathscr{C}) \colon \mathsf{Sh}(X, \mathscr{C}) \longrightarrow \mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C}), \tag{2.7.2}$$

defined on objects in the obvious way. Its actual functoriality is just a consequence of the definition of morphism of $\mathcal{B}$-sheaves, and is an easy routine check.

LEMMA 2.7.7. *A $\mathcal{B}$-sheaf $\{\mathcal{F}(\mathcal{B}), \rho_{\mathcal{B}}\}$ uniquely extends to a sheaf $\overline{\mathcal{F}}$, such that $\overline{\mathcal{F}}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$.*

Proof. Let $U \in \tau_X$ be an arbitrary open set. Define

$$\overline{\mathcal{F}}(U) = \left\{ \text{tuples } (s_x)_{x \in U} \text{ of } \mathcal{B}\text{-compatible germs} \right\} \subset \prod_{x \in U} \mathcal{F}_x.$$

This is manifestly a presheaf. It is also clear that the above definition agrees with $\mathcal{F}(U)$ whenever $U \in \mathcal{B}$, since the injective map $\sigma_U^{\mathcal{F}}$ hits precisely the tuples of $\mathcal{B}$-compatible germs; moreover, for the same reason, this definition is the *only* possible extension of the original $\mathcal{B}$-sheaf. The sheaf property is fulfilled by $\overline{\mathcal{F}}$ precisely for the same reason why it is fulfilled by the sheafification of a presheaf (see the proof of Proposition 2.5.2). □

In fact, the statement of Lemma 2.7.7 can be made functorial: one can prove that the restriction functor (2.7.2) is an equivalence. The inverse is given precisely by Lemma 2.7.7 above at the level of objects and by Proposition 2.7.9 below for morphisms.

Remark 2.7.8. We have that $\mathcal{F}_x = \overline{\mathcal{F}}_x$ for all $x \in X$. This follows directly from Remark 2.7.4.

The analogue of Lemma 2.7.7 for *morphisms* is the following.

PROPOSITION 2.7.9. *Let X be a topological space, $\mathcal{B} \subset \tau_X$ a base of open sets and $\mathcal{F}, \mathcal{G}$ two sheaves on X. Suppose given a morphism*

$$\eta_{\mathcal{B}} \colon \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{F}) \to \mathsf{res}_{\mathcal{B}}(X, \mathscr{C})(\mathcal{G})$$

[1]Also $\mathcal{B}$-presheaves form a category, but it is not as well-behaved as $\mathsf{Sh}_{\mathcal{B}}(X, \mathscr{C})$, and we do not need it, so we shall ignore it.

between the underlying $\mathcal{B}$-sheaves. Then $\eta_{\mathcal{B}}$ extends uniquely to a sheaf homomorphism $\eta\colon \mathcal{F} \to \mathcal{G}$. Furthermore, if η_U is surjective (or injective, or an isomorphism) for every $U \in \mathcal{B}$, then so is η.

Exercise 2.7.10. Prove Proposition 2.7.9 and deduce that the restriction functor (2.7.2) is an equivalence.

2.8 Pushforward, inverse image

In this section we learn how to "move" sheaves from a topological space X to another topological space Y, in the presence of a continuous map between the two spaces.

$$
\begin{array}{ccc}
\mathcal{F} & \rightsquigarrow & f_*\mathcal{F} \\
| & & | \\
X & \xrightarrow{f} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
f^{-1}\mathcal{G} & \leftsquigarrow & \mathcal{G} \\
| & & | \\
X & \xrightarrow{f} & Y
\end{array}
$$

2.8.1 Pushforward (or direct image)

Let $f\colon X \to Y$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a presheaf on X. The assignment

$$V \mapsto f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}V)$$

defines a presheaf $f_*\mathcal{F}$ on Y, called the *pushforward* (or *direct image*) of $\mathcal{F}$ by f. It is a sheaf as soon as $\mathcal{F}$ is, because if $V = \bigcup_{i\in I} V_i$ is an open covering of an open subset $V \subset Y$, then $f^{-1}V = \bigcup_{i\in I} f^{-1}(V_i)$ is an open covering of $f^{-1}V \subset X$.

Example 2.8.1. If X is arbitrary and $Y = \mathsf{pt}$, then $f_*\mathcal{F}(\mathsf{pt}) = \mathcal{F}(X)$, an object of $\mathscr{C}$. We will see in a minute that the direct image along any continuous map defines a functor. The direct image along the constant map $(X \to \mathsf{pt})_*\colon \mathsf{Sh}(X,\mathscr{C}) \to \mathscr{C}$ is also called the *global section functor*. If $\mathscr{C} = \mathsf{Mod}_A$, it is a left exact functor (you already proved a more general statement in Exercise 2.5.16).

Example 2.8.2. If $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then $f_*\mathcal{F}$ is defined, for any open subset $V \subset Y$, by

$$f_*\mathcal{F}(V) = \mathcal{F}(V \cap X).$$

Example 2.8.3 (Skyscraper sheaf as a pushforward)**.** Let $x \in X$ be a point, G a nontrivial abelian group. Consider the constant sheaf $\underline{G}_{\{x\}}$ on $\{x\}$. Let $i_x\colon \{x\} \hookrightarrow X$ be the inclusion. Then the skyscraper sheaf $G_x \in \mathsf{Sh}(X,\mathsf{Mod}_{\mathbb{Z}})$ defined in Example 2.4.12 can be described as

$$G_x = i_{x,*}\underline{G}_{\{x\}}.$$

Next, we observe that pushforward of sheaves is functorial, i.e. sending $\mathcal{F} \mapsto f_*\mathcal{F}$ defines functors

$$\begin{array}{ccc} \mathsf{Sh}(X,\mathscr{C}) & \xrightarrow{f_*} & \mathsf{Sh}(Y,\mathscr{C}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{pSh}(X,\mathscr{C}) & \xrightarrow{f_*} & \mathsf{pSh}(Y,\mathscr{C}) \end{array}$$

where the vertical maps are the natural inclusions (2.2.2). Indeed, given a morphism of (pre)sheaves $\eta\colon \mathcal{F} \to \mathcal{G}$, we can construct a morphism of (pre)sheaves

$$f_*\eta\colon f_*\mathcal{F} \to f_*\mathcal{G}$$

simply by setting

$$(f_*\eta)_V = \eta_{f^{-1}V} : \mathcal{F}(f^{-1}V) \to \mathcal{G}(f^{-1}V)$$

for an open subset $V \subset Y$. The compatibility with restriction maps follows from those of η (and the obvious observation that if $V' \subset V$ then $f^{-1}V' \subset f^{-1}V$).

Moreover, $(-)_*$ is compatible with compositions of continuous maps, in the following sense: if $f: X \to Y$ and $g: Y \to Z$ are continuous maps of topological spaces, then, as functors, we have an equality $(g \circ f)_* = g_* \circ f_*$ on the nose (both for presheaves and for sheaves). In other words, the diagram

(2.8.1)
$$\begin{array}{ccc} \mathsf{Sh}(X,\mathscr{C}) & \xrightarrow{f_*} & \mathsf{Sh}(Y,\mathscr{C}) \\ & {\scriptstyle (g\circ f)_*} \searrow & \big\downarrow {\scriptstyle g_*} \\ & & \mathsf{Sh}(Z,\mathscr{C}) \end{array}$$

commutes. Indeed, if $\mathcal{F}$ is a (pre)sheaf on X, then for every open $W \subset Z$ one has

$$\begin{aligned} (g \circ f)_*\mathcal{F}(W) &= \mathcal{F}((g \circ f)^{-1}(W)) \\ &= \mathcal{F}(f^{-1}g^{-1}(W)) \\ &= f_*\mathcal{F}(g^{-1}(W)) \\ &= (g_*f_*\mathcal{F})(W) \\ &= (g_* \circ f_*)\mathcal{F}(W). \end{aligned}$$

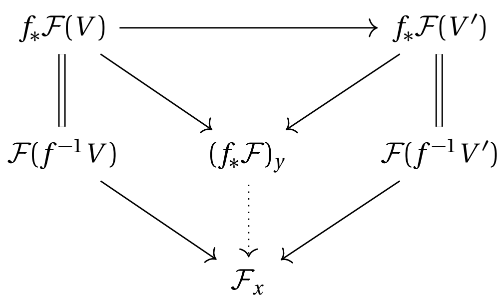
Note that no identifications are made here: all equalities are actual equalities!

LEMMA 2.8.4. *Let $f: X \to Y$ be a continuous map of topological spaces, and fix a sheaf $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. Let $x \in X$ be a point, and set $y = f(x)$. There is a canonical morphism*

$$(f_*\mathcal{F})_y \longrightarrow \mathcal{F}_x,$$

which is an isomorphism when f is the inclusion of a subspace $X \hookrightarrow Y$.

Proof. If $y \in V' \subset V \subset Y$, then $x \in f^{-1}V' \subset f^{-1}V \subset X$, and the commutative diagram



induces, via the universal property of the stalk (cf. Definition B.3.6)

$$(f_*\mathcal{F})_y = \varinjlim_{V \ni y} \mathcal{F}(f^{-1}V),$$

a canonical morphism $(f_*\mathcal{F})_y \to \mathcal{F}_x$, as required.

Now, let us assume $f \colon X \hookrightarrow Y$ is the inclusion of a subspace, and let us take $y \in X$. Note that every neighbourhood $y \in U \subset X$ is of the form $U = V \cap X$ for some open neighbourhood $y \in V \subset Y$. Thus

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X) \xrightarrow{\sim} \varinjlim_{X \supset U \ni y} \mathcal{F}(U) = \mathcal{F}_y. \tag{2.8.2}$$

The proof is complete. □

Remark 2.8.5. We shall use Lemma 2.8.4 crucially with $\mathscr{C}$ = Rings, when defining morphisms of locally ringed spaces (cf. Remark 2.10.5).

Caution 2.8.6. Even if $f \colon X \hookrightarrow Y$ is the inclusion of a subspace, it is not true that $(f_*\mathcal{F})_y = 0$ for all $y \in Y \setminus X$. This is nevertheless true when f is the inclusion of a *closed* subspace, cf. Remark 2.8.7.

Remark 2.8.7. If $f \colon X \hookrightarrow Y$ is the inclusion of a *closed* subspace, and $\mathcal{F}$ is a sheaf on X, then

$$(f_*\mathcal{F})_y = \begin{cases} \mathcal{F}_y & \text{if } y \in X \\ 0 & \text{if } y \notin X. \end{cases} \tag{2.8.3}$$

The case $y \in X$ is the computation (2.8.2). As for the case $y \notin X$, we use the definition

$$(f_*\mathcal{F})_y = \varinjlim_{Y \supset V \ni y} f_*\mathcal{F}(V) = \varinjlim_{Y \supset V \ni y} \mathcal{F}(V \cap X),$$

and the observation that, since X is closed in Y, there are arbitrarily small neighbourhoods V of y which are disjoint from X. For these, we have $\mathcal{F}(V \cap X) = \mathcal{F}(\emptyset) = 0$ since $\mathcal{F}$ is a sheaf (Remark 2.3.2). This causes the colimit to vanish.

Exactness of pushforward

We set $\mathscr{C} = \mathsf{Mod}_A$ in this subsection (for A a fixed ring), and we fix a continuous map $f: X \to Y$. Consider the direct image functor

$$f_*\colon \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A).$$

It is important to remember that

$$\boxed{f_* \text{ is always left exact, and it is exact if } f: X \hookrightarrow Y \text{ is a closed subspace.}}$$

Since f_* will turn out to be a right adjoint (Lemma 2.8.16), it is left exact by general category theory. However, we prove it directly here. Note that you have already proved the case $Y = \mathrm{pt}$ in Exercise 2.5.16. You will notice in the proof of the above slogan that this was essentially enough to handle the general case.

PROPOSITION 2.8.8. *Let A be a ring, $f: X \to Y$ a continuous map of topological spaces. The functor $f_*\colon \mathsf{Sh}(X, \mathsf{Mod}_A) \to \mathsf{Sh}(Y, \mathsf{Mod}_A)$ is left exact. If f is the inclusion of a closed subspace, then f_* is exact.*

Proof. Let us prove the first assertion. We have to show that an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\ \alpha\ } \mathcal{G} \xrightarrow{\ \beta\ } \mathcal{H}$$

in $\mathsf{Sh}(X, \mathsf{Mod}_A)$ induces an exact sequence

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{\ f_*\alpha\ } f_*\mathcal{G} \xrightarrow{\ f_*\beta\ } f_*\mathcal{H}$$

in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. We know by Exercise 2.5.16 that we have an exact sequence

$$0 \longrightarrow \mathcal{F}(f^{-1}V) \xrightarrow{\ \alpha_{f^{-1}V}\ } \mathcal{G}(f^{-1}V) \xrightarrow{\ \beta_{f^{-1}V}\ } \mathcal{H}(f^{-1}V) \tag{2.8.4}$$

for any open subset $V \subset Y$, by applying the functor $\Gamma(f^{-1}V, -)$ to the original sequence. In particular, $\alpha_{f^{-1}V} = (f_*\alpha)_V$ is injective for all V, which shows that $f_*\alpha$ is injective. There is an equality of presheaves

$$\mathrm{im}_{\mathrm{pre}}(f_*\alpha) = \ker(f_*\beta)$$

again thanks to exactness of (2.8.4) in the middle, ensuring precisely that $\mathrm{im}(\alpha_{f^{-1}V}) = \ker(\beta_{f^{-1}V})$. But $\ker(f_*\beta)$ is a sheaf, therefore we get exactness in the middle, i.e. $\mathrm{im}(f_*\alpha) = \ker(f_*\beta)$.

Let us show the second statement. Assume f is the inclusion of a closed subspace. By the first part of the proof, we only need to show that if $\eta\colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ is surjective as a map of sheaves on X, then $f_*\mathcal{G} \twoheadrightarrow f_*\mathcal{H}$ is surjective as a map of sheaves on Y. We check this on stalks. If $y \in Y \setminus X$, then (using that X is closed, cf. Remark 2.8.7)

$$(f_*\mathcal{G})_y = 0 = (f_*\mathcal{H})_y, \tag{2.8.5}$$

so there is nothing to prove here. Assume $y \in X$. Since $\mathcal{G}$ surjects onto $\mathcal{H}$, in the commutative diagram

$$\begin{array}{ccc} (f_*\mathcal{G})_y & \xrightarrow{(f_*\eta)_y} & (f_*\mathcal{H})_y \\ \| & & \| \\ \mathcal{G}_y & \xrightarrow{\eta_y} & \mathcal{H}_y \end{array}$$

the bottom map is surjective. The vertical equalities are given by Remark 2.8.7. Thus the top map is surjective as well. Hence $f_*\eta$ is surjective on all stalks, hence it is surjective. □

2.8.2 Inverse image

Let $f\colon X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Given $U \subset X$, the collection of open subsets $V \subset Y$ containing $f(U)$ form a directed set via reverse inclusions. Sending

$$U \mapsto (f^{-1}_{\text{pre}}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

defines a presheaf on X. Indeed, assume $U' \subset U$ is an open subset. Then there is an inclusion $f(U') \subset f(U)$, inducing a map of directed systems

$$\{\, V \in \tau_Y \mid V \supset f(U)\,\} \hookrightarrow \{\, V \in \tau_Y \mid V \supset f(U')\,\},$$

which in turn induces a morphism

$$(f^{-1}_{\text{pre}}\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V) \longrightarrow \varinjlim_{V \supset f(U')} \mathcal{G}(V) = (f^{-1}_{\text{pre}}\mathcal{G})(U').$$

This is the restriction morphism $\rho_{UU'}$ for $f^{-1}_{\text{pre}}\mathcal{G}$.

Remark 2.8.9. If $f(U)$ is an open subset of Y, then

$$(f^{-1}_{\text{pre}}\mathcal{G})(U) = \mathcal{G}(f(U)).$$

Now assume $\mathcal{G}$ is a sheaf. We define the *inverse image* of $\mathcal{G}$ by f to be the sheafification

$$f^{-1}\mathcal{G} = \left(f^{-1}_{\text{pre}}\mathcal{G}\right)^{\#}.$$

By Proposition 2.5.2, the canonical map $f^{-1}_{\text{pre}}\mathcal{G} \to f^{-1}\mathcal{G}$ of presheaves induces an isomorphism

$$(f^{-1}_{\text{pre}}\mathcal{G})_x \xrightarrow{\sim} (f^{-1}\mathcal{G})_x$$

on all the stalks.

Exercise 2.8.10. Both f^{-1}_{pre} and f^{-1} are functors.

Example 2.8.11. Let $\iota_y\colon \{y\} \hookrightarrow Y$ be the inclusion of a point $y \in Y$, and let $\mathcal{G}$ be a sheaf on Y. Then $\iota_y^{-1}\mathcal{G} = \mathcal{G}_y$, since $\iota_y^{-1}\mathcal{G}(\{y\}) = \varinjlim_{V\ni y} \mathcal{G}(V) = \mathcal{G}_y$. Thus ι_y^{-1} agrees with the stalk functor

$$\mathsf{stalk}_y\colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathscr{C}, \quad \mathcal{G} \mapsto \mathcal{G}_y.$$

Example 2.8.12. If $p\colon X \to \mathsf{pt}$ is the constant map, and $G \in \mathscr{C} \cong \mathsf{Sh}(\mathsf{pt},\mathscr{C})$, then $p^{-1}G = \underline{G}_X$, the constant sheaf on X with values in the object G.

Example 2.8.13. Let $j\colon U \hookrightarrow Y$ be the inclusion of an open subset. Then $j_{\mathrm{pre}}^{-1}\mathcal{G} = \mathcal{G}|_U$ for any sheaf $\mathcal{G}$ on Y. The reason is that if U' is open in U, it is also open in Y, and thus

$$j_{\mathrm{pre}}^{-1}\mathcal{G}(U') = \varinjlim_{V\supset U'} \mathcal{G}(V) = \mathcal{G}(U') = \mathcal{G}|_U(U').$$

In particular, $j_{\mathrm{pre}}^{-1}\mathcal{G}$ is already a sheaf, and hence

$$j^{-1}\mathcal{G} = \mathcal{G}|_U, \quad U \subset Y \text{ open}.$$

Remark 2.8.14. Despite Example 2.8.13, sheafifying f_{pre}^{-1} is in general necessary: consider a constant map $f\colon X = \{\star,\bullet\} \to \{*\} = Y$ from a two point space, and fix a nontrivial abelian group G. The constant sheaf $\mathcal{G} = \underline{G}_Y$ has the property $f_{\mathrm{pre}}^{-1}\mathcal{G} = \underline{G}_X^{\mathrm{pre}}$, which is not a sheaf (cf. Example 2.2.18).

Functoriality (cf. Exercise 2.8.10) can be translated into a diagram of functors

$$\begin{array}{ccc} \mathsf{Sh}(Y,\mathscr{C}) & \xrightarrow{f^{-1}} & \mathsf{Sh}(X,\mathscr{C}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{pSh}(Y,\mathscr{C}) & \xrightarrow{f_{\mathrm{pre}}^{-1}} & \mathsf{pSh}(X,\mathscr{C}) \end{array}$$

where f^{-1} is obtained by applying $(-)^{\#}\colon \mathsf{pSh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$ in the last step.

Exercise 2.8.15. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of topological spaces. Show that

$$f^{-1} \circ g^{-1} = (g \circ f)^{-1}$$

as functors $\mathsf{Sh}(Z,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$.

LEMMA 2.8.16 (Unit and counit maps). *For any pair of presheaves* $\mathcal{F} \in \mathsf{pSh}(X,\mathscr{C})$ *and* $\mathcal{G} \in \mathsf{pSh}(Y,\mathscr{C})$ *there are canonical presheaf homomorphisms*

$$\mathcal{G} \xrightarrow{\ \mathsf{unit}\ } f_* f_{\mathrm{pre}}^{-1}\mathcal{G}, \qquad f_{\mathrm{pre}}^{-1} f_*\mathcal{F} \xrightarrow{\ \mathsf{counit}\ } \mathcal{F}.$$

Proof. We start with the unit map. The observation here is that there is, for any open subset $V \subset Y$, a canonical inclusion $f(f^{-1}V) \subset V$. Thus $\mathcal{G}(V)$ appears in the colimit

$$\varinjlim_{W\supset f(f^{-1}V)} \mathcal{G}(W).$$

This induces a canonical morphism

$$\mathsf{unit}_V : \mathcal{G}(V) \to \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) = f_{\mathrm{pre}}^{-1}\mathcal{G}(f^{-1}V) = f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V)$$

which does define a natural transformation $\mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ because if $V' \subset V$, then any open $W \subset Y$ containing $f(f^{-1}V)$ also contains $f(f^{-1}V')$, simply because $f(f^{-1}V') \subset f(f^{-1}V)$. Thus there is a natural morphism

$$\varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) \to \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W)$$

and the induced diagram

$$\begin{array}{ccccc} \mathcal{G}(V) & \xrightarrow{\mathsf{unit}_V} & \varinjlim_{W \supset f(f^{-1}V)} \mathcal{G}(W) & =\!=\!= & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V) \\ \downarrow & & \downarrow & & \\ \mathcal{G}(V') & \xrightarrow{\mathsf{unit}_{V'}} & \varinjlim_{W \supset f(f^{-1}V')} \mathcal{G}(W) & =\!=\!= & f_* f_{\mathrm{pre}}^{-1}\mathcal{G}(V') \end{array}$$

commutes. This defines the map $\mathsf{unit} \colon \mathcal{G} \to f_* f_{\mathrm{pre}}^{-1}\mathcal{G}$ of presheaves.

To construct the map counit, one observes that for any open subset $U \subset X$ there is (by the universal property of colimits, cf. Definition B.3.6) a canonical map

$$f_{\mathrm{pre}}^{-1} f_* \mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_* \mathcal{F}(V) = \varinjlim_{V \supset f(U)} \mathcal{F}(f^{-1}V) \to \mathcal{F}(U),$$

since if $V \supset f(U)$ inside Y, then $U \subset f^{-1}f(U) \subset f^{-1}V$ inside X. This map is also functorial in $U' \subset U$, thus the map $\mathsf{counit} \colon f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \to \mathcal{F}$ is defined. $\square$

The usefulness of the homomorphisms unit and counit is that they make $(f_{\mathrm{pre}}^{-1}, f_*)$ into an adjoint pair of functors. More precisely, there are bijections

$$\varphi_{\mathcal{F},\mathcal{G}} \colon \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f_{\mathrm{pre}}^{-1}\mathcal{G}, \mathcal{F}),$$

functorial in both $\mathcal{F}$ and $\mathcal{G}$. Specifically, $\varphi_{\mathcal{F},\mathcal{G}}$ sends $\eta \colon \mathcal{G} \to f_*\mathcal{F}$ to

$$f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_{\mathrm{pre}}^{-1}\eta} f_{\mathrm{pre}}^{-1} f_* \mathcal{F} \xrightarrow{\mathsf{counit}} \mathcal{F}$$

with inverse sending $\zeta \colon f_{\mathrm{pre}}^{-1}\mathcal{G} \to \mathcal{F}$ to

$$\mathcal{G} \xrightarrow{\mathsf{unit}} f_* f_{\mathrm{pre}}^{-1}\mathcal{G} \xrightarrow{f_*\zeta} f_*\mathcal{F}.$$

Using the adjunction

$$\mathsf{pSh}(Y,\mathscr{C}) \underset{f_*}{\overset{f_{\mathrm{pre}}^{-1}}{\rightleftarrows}} \mathsf{pSh}(X,\mathscr{C}) \tag{2.8.6}$$

it is immediate to show that also

$$\mathsf{Sh}(Y,\mathscr{C}) \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathsf{Sh}(X,\mathscr{C}) \tag{2.8.7}$$

is an adjoint pair of functors. Indeed, for any pair of sheaves $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$ and $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) &= \mathrm{Hom}_{\mathsf{pSh}(Y,\mathscr{C})}(\mathcal{G}, f_*\mathcal{F}) && j_{Y,\mathscr{C}} \text{ is fully faithful} \\ &\simeq \mathrm{Hom}_{\mathsf{pSh}(X,\mathscr{C})}(f^{-1}_{\mathrm{pre}}\mathcal{G}, \mathcal{F}) && \text{adjunction (2.8.6)} \\ &\simeq \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, \mathcal{F}) && \text{Exercise 2.5.5.} \end{aligned}$$

Remark 2.8.17. Fix $\mathcal{G} \in \mathsf{Sh}(Y,\mathscr{C})$. Once more, the adjunction (2.8.7) gives a canonical morphism $\mathcal{G} \to f_*f^{-1}\mathcal{G}$, corresponding to $\mathrm{id}_{f^{-1}\mathcal{G}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(\mathcal{G}, f_*f^{-1}\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}\mathcal{G}, f^{-1}\mathcal{G}).$$

Clearly sending $\mathcal{G} \mapsto f_*f^{-1}\mathcal{G}$ is a functor $f_*f^{-1}\colon \mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Rightarrow f_*f^{-1}$$

of functors $\mathsf{Sh}(Y,\mathscr{C}) \to \mathsf{Sh}(Y,\mathscr{C})$, which is called the *unit* of the adjunction (2.8.7). Similarly, let $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. There is a canonical morphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ corresponding to $\mathrm{id}_{f_*\mathcal{F}}$ under

$$\mathrm{Hom}_{\mathsf{Sh}(Y,\mathscr{C})}(f_*\mathcal{F}, f_*\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Sh}(X,\mathscr{C})}(f^{-1}f_*\mathcal{F}, \mathcal{F}).$$

Clearly sending $\mathcal{F} \mapsto f^{-1}f_*\mathcal{F}$ is a functor $f^{-1}f_*\colon \mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, and the naturality of this operation yields a natural transformation

$$\mathsf{counit}\colon f^{-1}f_* \Rightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

of functors $\mathsf{Sh}(X,\mathscr{C}) \to \mathsf{Sh}(X,\mathscr{C})$, which is called the *counit* of the adjunction (2.8.7).

The next lemma says that the stalk of the inverse image is somewhat easy to compute (unlike for the pushforward).

LEMMA 2.8.18 (Stalk of inverse image). *Let $f\colon X \to Y$ be a continuous map of topological spaces, $\mathcal{G}$ a sheaf on Y, and $x \in X$ a point. There is a canonical identification*

$$(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}.$$

Proof. We have

$$\begin{aligned}
(f^{-1}\mathcal{G})_x &= (\{x\} \hookrightarrow X)^{-1}(f^{-1}\mathcal{G}) && \text{by Example 2.8.11} \\
&= (\{x\} \hookrightarrow X \to Y)^{-1}\mathcal{G} && \text{by Exercise 2.8.15} \\
&= (\{f(x)\} \hookrightarrow Y)^{-1}\mathcal{G} \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

where we have used Example 2.8.11 once more for the last identity. □

Exercise 2.8.19. Show that if $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1}f_*\mathcal{F} \to \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \mathsf{Sh}(X,\mathscr{C})$. (**Hint**: check it on stalks).

PROPOSITION 2.8.20. *Let $f\colon X \hookrightarrow Y$ be the inclusion of a closed subspace.*

(1) *Let $\mathcal{G}$ be a sheaf on Y such that* $\operatorname{Supp}(\mathcal{G}) = X$. *Then the unit map*

$$\mathcal{G} \xrightarrow{\sim} f_*f^{-1}\mathcal{G}$$

is an isomorphism.

(2) *The functor f_* induces an equivalence of categories*

$$f_*\colon \mathsf{Sh}(X,\mathsf{Mod}_A) \xrightarrow{\sim} \mathsf{Sh}_X(Y,\mathsf{Mod}_A),$$

where $\mathsf{Sh}_X(Y,\mathsf{Mod}_A) \hookrightarrow \mathsf{Sh}(Y,\mathsf{Mod}_A)$ *is the full subcategory of sheaves on Y with support equal to X.*

Proof. To prove (1) it is enough to prove that the unit map is an isomorphism on all the stalks. If $y \in Y \setminus X$, we get $0 \xrightarrow{\sim} 0$, since $\mathcal{G}_y = 0$ by the assumption $\operatorname{Supp}(\mathcal{G}) = X$ and $(f_*f^{-1}\mathcal{G})_y = 0$ by (2.8.3). On the other hand, if $y \in X$, then $\mathcal{G}_y \to (f_*f^{-1}\mathcal{G})_y$ is nothing but the inverse of the isomorphism

$$(f_*f^{-1}\mathcal{G})_y \xrightarrow{\sim} (f^{-1}\mathcal{G})_y = \mathcal{G}_y$$

of Lemma 2.8.4.

To prove (2), observe first of all that f_* lands in the category $\mathsf{Sh}_X(Y,\mathsf{Mod}_A)$ by (2.8.3). Next, note that sending $\mathcal{G} \mapsto f^{-1}\mathcal{G}$ is an inverse to f_* by (1). In a little more detail, the equivalence (cf. Definition A.1.12) is set up by considering the pair of functors (f_*, f^{-1}) and exploiting the unit and counit *natural isomorphisms*

$$\text{unit}\colon \mathrm{Id}_{\mathsf{Sh}(Y,\mathscr{C})} \Longrightarrow f_*f^{-1} \qquad \text{counit}\colon f^{-1}f_* \Longrightarrow \mathrm{Id}_{\mathsf{Sh}(X,\mathscr{C})}$$

using (1) and Exercise 2.8.19. □

Exercise 2.8.21. Find examples of maps f and sheaves $\mathcal{G}$ such that $\mathcal{G} \to f_* f^{-1}\mathcal{G}$ is not an isomorphism.

Remark 2.8.22. If $j: X \hookrightarrow Y$ is *open* and $\mathcal{G}$ is a sheaf on Y, then $j_* j^{-1}\mathcal{G}$ satisfies

$$j_* j^{-1}\mathcal{G}(V) = (j_*\mathcal{G}|_X)(V) = \mathcal{G}(V \cap X), \quad V \subset Y \text{ open}.$$

The natural map $\mathcal{G}(V) \to j_* j^{-1}\mathcal{G}(V)$ sends $s \mapsto s|_{V\cap X}$.

PROPOSITION 2.8.23. *Let $\mathscr{C} = \mathsf{Mod}_A$, for a ring A. Then the inverse image functor*

$$f^{-1}: \mathsf{Sh}(Y, \mathsf{Mod}_A) \to \mathsf{Sh}(X, \mathsf{Mod}_A)$$

is exact, for any continuous map $f: X \to Y$ of topological spaces.

Proof. Indeed, let

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow \mathcal{K} \longrightarrow 0$$

be an exact sequence in $\mathsf{Sh}(Y, \mathsf{Mod}_A)$. Then,

$$0 \longrightarrow \mathcal{G}_{f(x)} \longrightarrow \mathcal{H}_{f(x)} \longrightarrow \mathcal{K}_{f(x)} \longrightarrow 0$$

is exact in Mod_A by Proposition 2.5.14, for every $x \in X$. But by Lemma 2.8.18, this is precisely the sequence

$$0 \longrightarrow (f^{-1}\mathcal{G})_x \longrightarrow (f^{-1}\mathcal{H})_x \longrightarrow (f^{-1}\mathcal{K})_x \longrightarrow 0.$$

Thus

$$0 \longrightarrow f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{H} \longrightarrow f^{-1}\mathcal{K} \longrightarrow 0$$

is exact, again by Proposition 2.5.14. □

2.9 Gluing sheaves

The purpose of this section is to prove the next theorem, which is of crucial importance (see e.g. the proof of Theorem 3.2.13).

THEOREM 2.9.1 (Gluing sheaves). *Let X be a topological space, $X = \bigcup_{i\in I} U_i$ an open covering. Set $U_{ij} = U_i \cap U_j$ and similarly $U_{ijk} = U_{ij} \cap U_k$. Assume given a sheaf $\mathcal{F}_i$ on U_i for every $i \in I$, along with a collection of isomorphisms*

$$\varphi_{ij}: \mathcal{F}_i\big|_{U_{ij}} \xrightarrow{\sim} \mathcal{F}_j\big|_{U_{ij}}$$

such that $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$ *for every* i*, and such that*

$$\begin{array}{ccc} \mathcal{F}_i\big|_{U_{ijk}} & \xrightarrow{\varphi_{ij}|_{U_{ijk}}} & \mathcal{F}_j\big|_{U_{ijk}} \\ & \searrow^{\varphi_{ik}|_{U_{ijk}}} & \downarrow{\scriptstyle \varphi_{jk}|_{U_{ijk}}} \\ & & \mathcal{F}_k\big|_{U_{ijk}} \end{array}$$

commutes for every triple intersection. Then there is a unique sheaf $\mathcal{F}$ *on* X *equipped with isomorphisms*

$$\alpha_i\colon \mathcal{F}\big|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$$

such that the diagrams

$$\begin{array}{ccccc} \mathcal{F}\big|_{U_{ij}} & & \xrightarrow{\alpha_j|_{U_{ij}}} & & \mathcal{F}_j\big|_{U_{ij}} \\ & \searrow^{\alpha_i|_{U_{ij}}} & & \nearrow^{\varphi_{ij}} & \\ & & \mathcal{F}_i\big|_{U_{ij}} & & \end{array}$$

commute for every $(i,j) \in I \times I$*. The sheaf* $\mathcal{F}$ *is called the gluing of* $(\mathcal{F}_i, \varphi_{ij})_{i,j}$ *along the given covering.*

2.10 Locally ringed spaces

Given our background on sheaves, we are ready for the definition of locally ringed space.

Definition 2.10.1 (Locally ringed space)**.** A *ringed space* is a pair $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a sheaf of commutative rings on X. The sheaf $\mathscr{O}_X$ is called the *structure sheaf*. A *locally ringed space* is a ringed space such that the stalk $\mathscr{O}_{X,x}$ is a local ring for every $x \in X$.

Notation 2.10.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space, $x \in X$ a point. We will write $\mathfrak{m}_x$ for the maximal ideal $\mathscr{O}_{X,x}$, and $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ for the corresponding *residue field*.

Recall that, given two local rings $(B, \mathfrak{m}_B)$ and $(A, \mathfrak{m}_A)$, a local homomorphism between them is a ring homomorphism $h\colon B \to A$ such that $h^{-1}(\mathfrak{m}_A) = \mathfrak{m}_B$, or, equivalently, $h(\mathfrak{m}_B) \subset \mathfrak{m}_A$.

Definition 2.10.3 (Morphism of locally ringed spaces)**.** A morphism of locally ringed spaces, denoted

$$(X, \mathscr{O}_X) \xrightarrow{(f, f^{\#})} (Y, \mathscr{O}_Y), \tag{2.10.1}$$

is a pair $(f, f^{\#})$ where $f\colon X \to Y$ is a continuous map between the underlying topological spaces and $f^{\#}\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ is a sheaf homomorphism on Y, such that $f^{\#}_x\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism of local rings for every $x \in X$.

Notation 2.10.4. In what follows, when there is no confusion possible, we shall omit the sheaf of rings from the notation, and simply write X to denote the locally ringed space $(X, \mathscr{O}_X)$, or $f: X \to Y$ to denote a morphism $(f, f^\#)$ of locally ringed spaces as in (2.10.1). When we want to emphasise the underlying topological space of $(X, \mathscr{O}_X)$, we write $|X|$.

Remark 2.10.5. Let $f: X \to Y$ be a morphism of locally ringed spaces. Let $x \in X$ be a point, and set $y = f(x)$. The local homomorphism $f_x^\#: \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is the composition of the stalk map $f_y^\#: \mathscr{O}_{Y,y} \to (f_* \mathscr{O}_X)_y$ and the morphism $(f_* \mathscr{O}_X)_y \to \mathscr{O}_{X,x}$ of Lemma 2.8.4.

Example 2.10.6. Let $(X, \mathscr{O}_X)$ be a locally ringed space, $U \subset X$ an open subset. Then $(U, \mathscr{O}_X|_U)$ is a locally ringed space. We shall *always* take $\mathscr{O}_X|_U$ as the structure sheaf of an open subset $U \subset X$ of a locally ringed space X. We denote it by $\mathscr{O}_U$.

The composition of two morphisms of locally ringed spaces is defined in a straightforward way (but you need to know that pushforward commutes with composition, see (2.8.1)). In a little more detail, consider two morphisms

$$(X, \mathscr{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathscr{O}_Y), \quad (Y, \mathscr{O}_Y) \xrightarrow{(g, g^\#)} (Z, \mathscr{O}_Z)$$

and define their composition to be the morphism

$$(X, \mathscr{O}_X) \xrightarrow{(g \circ f, (g \circ f)^\#)} (Z, \mathscr{O}_Z)$$

where the map on sheaves $(g \circ f)^\#$ is the composition

$$\mathscr{O}_Z \xrightarrow{g^\#} g_* \mathscr{O}_Y \xrightarrow{g_* f^\#} g_* f_* \mathscr{O}_X = (g \circ f)_* \mathscr{O}_X.$$

Locally ringed spaces thus form a (large) category, denoted

$$\mathsf{LRS},$$

where isomorphisms are simply the invertible morphisms (those admitting a morphism in the opposite direction such that compositions are the identity both ways).

Remark 2.10.7. A morphism of locally ringed spaces $(f, f^\#)$ as in (2.10.1) is an isomorphism if and only if

- $f: X \to Y$ is a topological homeomorphism, and
- $f^\#: \mathscr{O}_Y \to f_* \mathscr{O}_X$ is an isomorphism of sheaves.

Definition 2.10.8 (Immersions)**.** Let $(f, f^\#): (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. It is called an *open immersion* (resp. a *closed immersion*) if $f: X \to Y$ is a topological open immersion (resp. closed immersion) and $f_x^\#$ is an isomorphism (resp. surjective) for every $x \in X$. It is called an *immersion* (or a *locally closed immersion*) if it factors as a closed immersion followed by an open immersion.

Notation 2.10.9. We sometimes may, and will, denote an immersion by '$X \hookrightarrow Y$'.

It is clear that a morphism of locally ringed spaces $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is an open immersion if and only if there exists an open subset $V \subset Y$ such that $(f, f^\#)$ induces an isomorphism $(X, \mathscr{O}_X) \widetilde{\to} (V, \mathscr{O}_Y|_V)$. It is also clear that the composition of two open (resp. closed) immersions is an open immersion (resp. a closed immersion).

2.10.1 Closed immersions and ideal sheaves

In this section we characteriste closed immersions up to isomorphism by means *ideal sheaves*.

Definition 2.10.10 (Ideal sheaf)**.** Fix a locally ringed space $(X, \mathscr{O}_X)$. An *ideal sheaf* (or a *sheaf of ideals*) is a subsheaf $\mathscr{I} \subset \mathscr{O}_X$ (as abelian groups) such that $\mathscr{I}(U) \subset \mathscr{O}_X(U)$ is an ideal for every open subset $U \subset X$.

Given an ideal sheaf $\mathscr{I}$, the subset

$$\mathrm{V}(\mathscr{I}) = \left\{ x \in X \,\middle|\, \mathscr{I}_x \neq \mathscr{O}_{X,x} \right\} \xhookrightarrow{\ j\ } X \tag{2.10.2}$$

is a closed subset. Indeed, for any $x \in X \setminus \mathrm{V}(\mathscr{I})$, i.e. for any x such that $\mathscr{I}_x = \mathscr{O}_{X,x}$, there is a neighbourhood U of x and a section $f \in \mathscr{I}(U)$ such that $f_x = 1 \in \mathscr{O}_{X,x}$. But this means that $f|_V = 1 \in \mathscr{O}_X(V)$ for some open subset $V \subset U$. Thus $V \subset X \setminus \mathrm{V}(\mathscr{I})$, and thus $X \setminus \mathrm{V}(\mathscr{I})$ is open.

The quotient sheaf $\mathscr{O}_X/\mathscr{I}$ is a sheaf of rings (because, by definition, it is the sheafification of a presheaf of rings), not just abelian groups. The pair

$$(\mathrm{V}(\mathscr{I}),\, j^{-1}(\mathscr{O}_X/\mathscr{I}))$$

defines a locally ringed space (indeed, for any $x \in \mathrm{V}(\mathscr{I})$, the stalk $(j^{-1}(\mathscr{O}_X/\mathscr{I}))_x = (\mathscr{O}_X/\mathscr{I})_{j(x)} = \mathscr{O}_{X,j(x)}/\mathscr{I}_{j(x)}$ is a local ring: we have used Lemma 2.8.18 and Remark 2.5.11), and the canonical surjection

$$j^\#\colon \mathscr{O}_X \twoheadrightarrow \mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$$

turns $(j, j^\#)$ into a closed immersion

$$(\mathrm{V}(\mathscr{I}),\, j^{-1}(\mathscr{O}_X/\mathscr{I})) \xhookrightarrow{(j,j^\#)} (X, \mathscr{O}_X). \tag{2.10.3}$$

Note that we have used Proposition 2.8.20 (1) for the identification $\mathscr{O}_X/\mathscr{I} = j_* j^{-1}(\mathscr{O}_X/\mathscr{I})$. So we have defined an assignment

$$\mathscr{O}_X \supset \mathscr{I} \qquad \longmapsto \qquad \mathrm{V}(\mathscr{I}) \hookrightarrow X.$$

Conversely, to any closed immersion $\iota\colon Y \hookrightarrow X$ one can associate an ideal sheaf, namely

$$\mathscr{I}_Y = \ker(\mathscr{O}_X \xrightarrow{\iota^\#} \iota_*\mathscr{O}_Y) \subset \mathscr{O}_X.$$

These two operations are inverse to each other "up to isomorphism", as the next proposition clarifies.

PROPOSITION 2.10.11. *Let* $(\iota,\iota^\#)\colon (Y,\mathscr{O}_Y) \hookrightarrow (X,\mathscr{O}_X)$ *be a closed immersion of locally ringed spaces. Set* $\mathscr{I}_Y = \ker\iota^\# \subset \mathscr{O}_X$ *and consider the associated closed immersion*

$$(\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}_Y)) \xhookrightarrow{(j,j^\#)} (X,\mathscr{O}_X).$$

as in (2.10.3)*. There is a unique isomorphism of locally ringed spaces*

$$(g,g^\#)\colon (Y,\mathscr{O}_Y) \xrightarrow{\sim} (\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}))$$

such that $(\iota,\iota^\#) = (j,j^\#)\circ(g,g^\#)$*.*

Moreover, an inclusion of ideal sheaves $\mathscr{I}_2 \hookrightarrow \mathscr{I}_1 \hookrightarrow \mathscr{O}_X$ *determines, and is determined by (in the above sense) a chain of closed immersions* $\mathrm{V}(\mathscr{I}_1) \hookrightarrow \mathrm{V}(\mathscr{I}_2) \hookrightarrow X$*.*

Proof. The last statement is immediate. We thus only prove the first.

Let us use the shorthand notation

$$(Z,\mathscr{O}_Z) = (\mathrm{V}(\mathscr{I}_Y), j^{-1}(\mathscr{O}_X/\mathscr{I}_Y)).$$

where $j\colon Z \hookrightarrow X$ is the topological closed embedding first appeared in (2.10.2). We need to find the isomorphism g as in the statement. By Remark 2.8.7, we have

$$(\iota_*\mathscr{O}_Y)_x = \begin{cases} \mathscr{O}_{Y,y} & \text{if } x = \iota(y) \\ 0 & \text{if } x \notin \iota(Y). \end{cases} \tag{2.10.4}$$

Combine the exact sequence

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to \iota_*\mathscr{O}_Y \to 0$$

with Proposition 2.5.14 and Equation (2.10.4) to deduce that

$$\mathscr{I}_{Y,x} = \mathscr{O}_{X,x} \quad\iff\quad x \notin \iota(Y).$$

This shows we have a homeomorphism $g\colon Y \xrightarrow{\sim} Z$. There is a factorisation

$$\iota\colon Y \xrightarrow{g} Z \xhookrightarrow{j} X$$

as topological maps, and

$$j_*\mathscr{O}_Z = j_*j^{-1}(\mathscr{O}_X/\mathscr{I}_Y) \cong \mathscr{O}_X/\mathscr{I}_Y = \iota_*\mathscr{O}_Y = j_*g_*\mathscr{O}_Y$$

as sheaves of rings, the last identity being a consequence of the above factorisation and Diagram (2.8.1). Therefore, we have

$$\mathscr{O}_Z \cong j^{-1} j_* \mathscr{O}_Z = j^{-1} j_* g_* \mathscr{O}_Y \cong g_* \mathscr{O}_Y .$$

This extends g to the desired isomorphism $(g, g^\#)$. It is straightforward to verify the identity

$$(\iota, \iota^\#) = (j, j^\#) \circ (g, g^\#)$$

as morphisms of locally ringed spaces. □

Proposition 2.10.11 will be used to make sense of the definition of *closed subscheme* (cf. Definition 3.2.4).

3 | Schemes

The goal of this chapter is to introduce the category Aff of *affine schemes* and the larger category Sch of all *schemes*. They will arise as full subcategories

$$\mathsf{Aff} \subset \mathsf{Sch} \subset \mathsf{LRS}.$$

3.1 Affine schemes

Let $A \neq 0$ be a nonzero ring (commutative, with unit $1 \neq 0$). The set

$$\boxed{\operatorname{Spec} A = \{\, \mathfrak{p} \subset A \mid \mathfrak{p} \text{ is a prime ideal} \,\}}$$

is called the *prime spectrum* of A. For now, this is just a set. We will endow it with a topology (cf. Corollary 3.1.7) and with a sheaf of rings $\mathscr{O}_{\operatorname{Spec} A}$ having local rings as stalks (cf. Theorem 3.1.28), to obtain a locally ringed space. Such locally ringed space will be called an *affine scheme* (cf. Important Definition 3.1.2). General schemes are obtained by glueing affine schemes, just as a smooth manifold is obtained by glueing open subsets of $\mathbb{R}^m$.

Notation 3.1.1. We introduce the following notation, that will be used throughout: given a ring B, the spectrum

$$\mathbb{A}^n_B = \operatorname{Spec} B[x_1, \ldots, x_n]$$

will be called *affine n-space* over B. If $n = 1$ (resp. $n = 2, 3$), we speak of *affine line* (resp. *affine plane, affine space*) over B.

Before getting started, we recall a few basic tools from commutative algebra (already used in Chapter 1).

Radical of an ideal

Definition 3.1.2 (Radical of an ideal)**.** The *radical* of an ideal $I \subset A$ is the subset

$$\sqrt{I} = \left\{ a \in A \,\middle|\, a^r \in I \text{ for some } r > 0 \right\} \subset A.$$

An ideal I is *radical* if $I = \sqrt{I}$.

Clearly $\sqrt{I} \subset A$ is an ideal containing I, and satisfies

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \mathfrak{p} \supset I}} \mathfrak{p}. \tag{3.1.1}$$

This implies that a prime ideal contains I if and only if it contains $\sqrt{I}$.

Remark 3.1.3. A prime ideal $\mathfrak{p} \subset A$ is radical (reason: let $a \in A$ be an element such that $a^r \in \mathfrak{p}$, with r is minimal; if $r = 1$ we are done, otherwise either $a \in \mathfrak{p}$ or $a^{r-1} \in \mathfrak{p}$ but the latter is excluded by minimality of r, thus $a \in \mathfrak{p}$), but the converse is false. For instance if $A = \mathbb{Z}$ we have $\sqrt{m\mathbb{Z}} = n\mathbb{Z}$, where $n = \prod_{p \mid m} p$. Thus if m is a product of distinct primes then $m\mathbb{Z}$ is radical but not prime.

Definition 3.1.4 (Nilradical of a ring)**.** The *nilradical* of a ring A is the ideal of nilpotent elements (the radical of the trivial ideal), namely

$$\operatorname{Nil}(A) = \sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \subset A.$$

A ring A is *reduced* if $\operatorname{Nil}(A) = 0$, i.e. if it contains no nontrivial nilpotents.

Operations on ideals

Let $(I_\lambda)_{\lambda \in \Lambda}$ be an arbitrary family of ideals in A. Then the intersection $\bigcap_{\lambda \in \Lambda} I_\lambda \subset A$ is easily checked to be an ideal. Recall that the sum of ideals $\sum_{\lambda \in \Lambda} I_\lambda$ is, by definition, the ideal generated by (i.e. the smallest ideal containing) the union of the ideals in the family (which is not an ideal in general). It can be described set-theoretically as

$$\sum_{\lambda \in \Lambda} I_\lambda = \left\{ \sum_{\lambda \in F} a_\lambda i_\lambda \;\middle|\; a_\lambda \in A,\ i_\lambda \in I_\lambda,\ |F| < \infty \right\}. \tag{3.1.2}$$

If we have *finitely many* ideals $I_1, I_2, \ldots, I_m \subset A$, their product is the ideal generated by the products of the form $i_1 i_2 \cdots i_m$, where $i_k \in I_k$ for $k = 1, \ldots, m$. In symbols,

$$I_1 I_2 \cdots I_m = \left\{ \sum_{1 \le j \le p} i_1^{(j)} i_2^{(j)} \cdots i_m^{(j)} \;\middle|\; i_k^{(j)} \in I_k,\ p < \infty \right\}.$$

In general, we have $I_1 I_2 \cdots I_m \subset I_1 \cap I_2 \cap \cdots \cap I_m$, with equality when $I_k + I_h = A$ for any pair (k, h) such that $k \neq h$ (if $I_k + I_h = A$ we say that I_k and I_h are *comaximal*).

3.1.1 The Zariski topology on Spec A

For an arbitrary ideal $I \subset A$, set

$$\boxed{\mathrm{V}(I) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I \} \subset \operatorname{Spec} A.}$$

Note that there is a bijection

$$\mathrm{V}(I) \simeq \operatorname{Spec} A/I,$$

since (prime) ideals of A/I correspond precisely to (prime) ideals in A containing I.

If $I = (f) = fA \subset A$ (we will use both notations for principal ideals) for $f \in A$, simply write $\mathrm{V}(f)$ instead of $\mathrm{V}(I)$, and define

$$\boxed{\mathrm{D}(f) = \operatorname{Spec} A \setminus \mathrm{V}(f) = \left\{ \mathfrak{p} \in \operatorname{Spec} A \;\middle|\; f \notin \mathfrak{p} \right\}.}$$

Example 3.1.5. Let $\mathbf{k}$ be an algebraically closed field. If $f \in \mathbf{k}[x]$ is nonzero, then $\mathrm{D}(f)$ consists of those prime ideals $\mathfrak{p} \subset \mathbf{k}[x]$ such that $f \notin \mathfrak{p}$. One such ideal is the trivial ideal (0), and the other ideals $\mathfrak{p}$ with this property are all the ideals of the form $\mathfrak{p} = (x - a)$, for $a \in \mathbf{k}$, such that $f(a) \neq 0 \in \mathbf{k}$.

Note that, for any ring A, one has

$$\operatorname{Spec} A = \mathrm{D}(1), \quad \emptyset = \mathrm{D}(0).$$

LEMMA 3.1.6. *Let A be a ring.*

(1) *If $I, J \subset A$ are two ideals, then* $\mathrm{V}(I) \cup \mathrm{V}(J) = \mathrm{V}(I \cap J)$.

(2) *If $(I_\lambda)_{\lambda \in \Lambda}$ is an arbitrary family of ideals, then* $\bigcap_{\lambda \in \Lambda} \mathrm{V}(I_\lambda) = \mathrm{V}\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$.

(3) $\operatorname{Spec} A = \mathrm{V}(0)$ *and* $\emptyset = \mathrm{V}(1)$.

Proof. This is straightforward. However, here is the proof:

(1) If $\mathfrak{p} \subset A$ contains either I or J, then it contains the smaller ideal $I \cap J$, thus $\mathrm{V}(I) \cup \mathrm{V}(J) \subset \mathrm{V}(I \cap J)$. If $\mathfrak{p} \supset I \cap J$ but $\mathfrak{p} \not\supset I$, there is $i \in I$ such that $i \notin \mathfrak{p}$. If $j \in J$, then $ij \in I \cap J \subset \mathfrak{p}$, which implies $j \in \mathfrak{p}$ (because $\mathfrak{p}$ is prime), thus $J \subset \mathfrak{p}$. Therefore $\mathrm{V}(I \cap J) \subset \mathrm{V}(I) \cup \mathrm{V}(J)$.

(2) If $\mathfrak{p}$ contains the sum $\sum_\lambda I_\lambda$, then it contains each I_λ, therefore $\mathrm{V}\left(\sum_\lambda I_\lambda\right) \subset \bigcap_\lambda \mathrm{V}(I_\lambda)$. On the other hand, assume $\mathfrak{p} \supset I_\lambda$ for every index λ. Let $h = a_1 i_{\lambda_1} + \cdots + a_p i_{\lambda_p}$ as in Equation (3.1.2). Then $a_j i_{\lambda_j} \in \mathfrak{p}$ by assumption, thus $h \in \mathfrak{p}$ as well, i.e. $\mathfrak{p} \supset \sum_\lambda I_\lambda$.

(3) Every prime $\mathfrak{p} \subset A$ contains $0 \in A$. No prime ideal $\mathfrak{p} \subset A$ contains $1 \in A$ (here we use that $1 \neq 0$). □

COROLLARY 3.1.7. *There exists a unique topology on* $\operatorname{Spec} A$ *whose closed sets are of the form* $\mathrm{V}(I)$. *Moreover, the sets* $\mathrm{D}(f) \subset \operatorname{Spec} A$ *form a base of open sets for this topology (according to Definition 2.7.1).*

Proof. The first statement is clear from Lemma 3.1.6 and the definition of a topology. The second one follows from these observations:

(i) $\mathrm{D}(f_1)\cap\mathrm{D}(f_2)=\mathrm{D}(f_1f_2)$ for all $f_1,f_2\in A$, and

(ii) an open subset $\operatorname{Spec}A\setminus\mathrm{V}(I)$ can be written as $\bigcup_{f\in I}\mathrm{D}(f)$. □

For instance,

$$\operatorname{Spec}A=\mathrm{D}(1)=\operatorname{Spec}A\setminus\mathrm{V}(1)=\bigcup_{f\in A}\mathrm{D}(f). \tag{3.1.3}$$

Important Definition 3.1.1 (Zariski topology)**.** The topology on $\operatorname{Spec}A$ given by Corollary 3.1.7 is called the *Zariski topology*.

Terminology 3.1.8. We call $\mathrm{D}(f)$ a *principal open set* in $\operatorname{Spec}A$, and $\mathrm{V}(f)$ a *principal closed set* in $\operatorname{Spec}A$.

Convention 3.1.9. When thinking of $\operatorname{Spec}A$ as a topological space, it will *always* be endowed with the Zariski topology.

Let $\mathbb{F}$ be a field. Consider the ideals $I_r=(x^r)\subset\mathbb{F}[x]$ for all $r>0$. Then $\mathrm{V}(I_1)=\{\mathfrak{p}\subset\mathbb{F}[x]\mid\mathfrak{p}\supset(x)\}=\{(x)\}=\mathrm{V}(I_r)$ for every r. Thus it may happen that

$$\mathrm{V}(I)=\mathrm{V}(J),\quad\text{with } I\neq J.$$

In general, by Equation (3.1.1), we have the set-theoretic identity

$$\mathrm{V}(I)=\mathrm{V}(\sqrt{I})\subset\operatorname{Spec}A.$$

LEMMA 3.1.10. *Let $I,J\subset A$ be two ideals in a ring A. Then*

$$\mathrm{V}(I)\subset\mathrm{V}(J)\iff J\subset\sqrt{I}.$$

Proof. This is a again a rephrasing of the identity $\sqrt{I}=\bigcap_{\mathfrak{p}\in\mathrm{V}(I)}\mathfrak{p}$, cf. Equation (3.1.1). □

Example 3.1.11. Let $f,g\in A$. We have

$$\begin{aligned}\mathrm{D}(g)\subset\mathrm{D}(f)&\iff\mathrm{V}(f)\subset\mathrm{V}(g)\\&\iff g\in\sqrt{fA}.\end{aligned}$$

LEMMA 3.1.12 (Spec A is quasicompact). *Fix a subset $S=\{f_k\mid k\in K\}\subset A$. Then* $\operatorname{Spec}A=\bigcup_{k\in K}\mathrm{D}(f_k)$ *if and only if there is a finite subset $F\subset K$ such that one can write $1=\sum_{k\in F}a_kf_k$ for some nonzero elements $a_k\in A$.*

In particular, $\operatorname{Spec}A$ *equipped with the Zariski topology is quasicompact.*

Proof. Let $(-)^{\mathrm{c}}$ denote the complement of a subset of $\operatorname{Spec}A$. The union

$$\bigcup_{k\in K}\mathrm{D}(f_k)=\bigcup_{k\in K}\operatorname{Spec}A\setminus\mathrm{V}(f_k)=\left(\bigcap_{k\in K}\mathrm{V}(f_k)\right)^{\mathrm{c}}=\mathrm{V}\left(\sum_{k\in K}f_kA\right)^{\mathrm{c}}$$

equals $\operatorname{Spec} A$ if and only if

$$\mathrm{V}\left(\sum_{k\in K} f_k A\right) = \emptyset = \mathrm{V}(1),$$

which by Lemma 3.1.10 happens if and only if $\sqrt{\sum_{k\in K} f_k A} = (1) = A$, which in turn means that $A = \sum_{k\in K} f_k A$. The first assertion then follows from the definition of sum of ideals. The last sentence in the statement follows from the first, setting $S = A$ and using (3.1.3). □

Remark 3.1.13. The proof of Lemma 3.1.12 also shows that any principal open subset $\mathrm{D}(f) \subset \operatorname{Spec} A$ is quasicompact, for

$$\mathrm{D}(f) = \mathrm{D}(f) \cap \operatorname{Spec} A = \mathrm{D}(f) \cap \bigcup_{k\in F} \mathrm{D}(f_k) = \bigcup_{k\in F} \mathrm{D}(f) \cap \mathrm{D}(f_k) = \bigcup_{k\in F} \mathrm{D}(f f_k).$$

Moreover, it shows that an open subset $U \subset \operatorname{Spec} A$ is quasicompact if and only if it is a finite union of principal opens.

Warning 3.1.14. Not *every* open subset $U \subset \operatorname{Spec} A$ is quasicompact! For instance, consider $A = \mathbf{k}[x_i \mid i \in \mathbb{N}]$, and let $U \subset \operatorname{Spec} A$ be the complement of the origin (the point corresponding to the maximal ideal $(x_i \mid i \in \mathbb{N}) \subset A$). Then the covering $U = \bigcup_{i\in\mathbb{N}} U \setminus \mathrm{V}(x_i)$ has no finite subcover.

Remark 3.1.15 (Closed points = maximal ideals)**.** Let $\mathfrak{p} \in \operatorname{Spec} A$ be a closed point, i.e. such that $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ is closed. Then $\{\mathfrak{p}\} = \mathrm{V}(I) = \{\mathfrak{q} \mid \mathfrak{q} \supset I\}$ for an ideal $I \subset A$. This says that $\mathfrak{p}$ is the only prime ideal containing I. But any ideal sits inside a maximal ideal, and maximal ideals are prime. Thus $\mathfrak{p}$ is maximal. Conversely, if $\mathfrak{m} \subset A$ is maximal, then $\{\mathfrak{m}\} = \mathrm{V}(\mathfrak{m})$, in particular $\{\mathfrak{m}\} \subset \operatorname{Spec} A$ is closed, i.e. $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point.

The previous remark can be generalised by the following lemma.

LEMMA 3.1.16. *Let $T \subset \operatorname{Spec} A$ be a subset, $\overline{T} \subset \operatorname{Spec} A$ its closure. Then*

$$\overline{T} = \mathrm{V}\left(\bigcap_{\mathfrak{p}\in T} \mathfrak{p}\right).$$

In particular, the closure of $\{\mathfrak{p}\} \subset \operatorname{Spec} A$ is precisely $\mathrm{V}(\mathfrak{p})$.

Proof. We have

$$\overline{T} = \bigcap_{\mathrm{V}(I)\supset T} \mathrm{V}(I) = \mathrm{V}\left(\sum_{\mathrm{V}(I)\supset T} I\right).$$

But by definition $\mathrm{V}(I) \supset T$ means that every $\mathfrak{p} \in T$ satisfies $\mathfrak{p} \supset I$, so the sum is over all $I \subset A$ such that $I \subset \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. The largest such ideal is precisely $\bigcap_{\mathfrak{p}\in T} \mathfrak{p}$, which concludes the proof. □

Remark 3.1.17. Combining the previous topological observations, we conclude that

the Zariski topology on $\operatorname{Spec} A$ is almost never Hausdorff, or even T_1.

For instance, take an integral domain A that is not a field, so that $(0) \subset A$ is prime and the closure of the corresponding point $\xi \in \operatorname{Spec} A$ is equal to $V(0) = \operatorname{Spec} A$ by Lemma 3.1.16. Then any two nonempty open subsets intersect (thus $\operatorname{Spec} A$ is not Hausdorff), and in fact every open neighbourhood of a point $\mathfrak{p} \in \operatorname{Spec} A$ will also contain ξ (thus $\operatorname{Spec} A$ is not T_1).

3.1.2 Interlude: functions on Spec A

The following slogan is important (and will be formalised in Theorem 3.1.28(b)):

elements of A are functions on $\operatorname{Spec} A$.

The slogan is a bit premature, since by 'function on $\operatorname{Spec} A$' we actually mean 'regular function on *the scheme* $\operatorname{Spec} A$', and so far we only have a topological space, we haven't yet defined the sheaf of regular functions. However, it is worth explaining the slogan just to build some intuition.

Here is the explanation. To any $f \in A$, we can associate the map

$$\theta_f \colon \operatorname{Spec} A \to \coprod_{\mathfrak{p} \in \operatorname{Spec} A} A/\mathfrak{p}, \quad \mathfrak{p} \mapsto f \bmod \mathfrak{p}.$$

For instance, $f = 9 \in \mathbb{Z}$ takes the value $[1]$ in $\mathbb{Z}/2\mathbb{Z}$, and the value $[4]$ in $\mathbb{Z}/5\mathbb{Z}$. Its value in $\mathbb{Z}/0 = \mathbb{Z}$ is just… 9.[1] Of course, the most confusing thing here is that the ring where the function takes values depends on the point on which the function is evaluated! Now obviously the function '9' vanishes on the point $(3) \in \operatorname{Spec} \mathbb{Z}$. In general,

$\theta_f(\mathfrak{p}) = 0 \in A/\mathfrak{p}$ if and only if $f \in \mathfrak{p}$.

Note also that addition and multiplication of 'functions' works as one might expect, i.e. $\theta_{f+g}(\mathfrak{p}) = f + g \bmod \mathfrak{p} = \theta_f(\mathfrak{p}) + \theta_g(\mathfrak{p})$, and similarly $\theta_{fg}(\mathfrak{p}) = fg \bmod \mathfrak{p} = \theta_f(\mathfrak{p})\theta_g(\mathfrak{p})$. This is just a rephrasing of the fact that $A \to A/\mathfrak{p}$ is a ring homomorphism!

Example 3.1.18. Consider $\mathbb{C}[x]$, and the 'function' $f(x) = 2x^2 - x + 3 \in \mathbb{C}[x]$. The prime ideals of $\mathbb{C}[x]$ are $(0) \subset \mathbb{C}[x]$, and the maximal ideals $\mathfrak{m}_a = (x - a) \subset \mathbb{C}[x]$ for $a \in \mathbb{C}$. The value of f on the point $\mathfrak{m}_a \in \operatorname{Spec} \mathbb{C}[x]$ is just the evaluation of the polynomial $f(x)$ at $x = a$. Indeed,

$$\theta_f(\mathfrak{m}_a) = f \bmod \mathfrak{m}_a \in \mathbb{C}[x]/\mathfrak{m}_a,$$

corresponds to the element

$$f(a) = 2a^2 - a + 3 \in \mathbb{C} \cong \mathbb{C}[x]/\mathfrak{m}_a.$$

[1] Please come back here after reading about generic points in Section 3.1.6.

Example 3.1.19. Consider $A = \mathbf{k}[t]/t^2$, and let $\overline{t} \in A$ be the image of $t \in \mathbf{k}[t]$ in A. If we see $\overline{t}$ as a function on $\operatorname{Spec} \mathbf{k}[t]/t^2$, we see that $\theta_{\overline{t}}$ evaluated on the point $(\overline{t})$ gives $0 \in A/\overline{t}$, i.e. the *nonzero* element $\overline{t} \in A$ determines a function that *vanishes at every point* of $\operatorname{Spec} \mathbf{k}[t]/t^2$. Here we encounter for the first time one of the magic aspects of scheme theory:

functions are not determined by their values on points!

This is due to the presence of nilpotents, which were not part of the game with classical algebraic varieties. As mentioned, we will see that $\operatorname{Spec} \mathbf{k}[t]/t^2 \neq \operatorname{Spec} \mathbf{k}$ *as affine schemes*, because their rings of functions are different: there is no ring isomorphism $\mathbf{k} \cong \mathbf{k}[t]/t^2$!

3.1.3 First examples of ring spectra

In this subsection we analyse the Zariski topology on $\operatorname{Spec} A$ for a few interesting rings A.

Example 3.1.20 (Spec $\mathbb{F}$, aka the point)**.** Let $\mathbb{F}$ be a field. The spectrum $\operatorname{Spec} \mathbb{F}$ consists of a single point corresponding to $(0) \subset \mathbb{F}$. Its 'functions' are just the constants $\mathbb{F}$, as expected. For now, this (merely topological and hence dry) description is enough. However, when $\operatorname{Spec} \mathbb{F}$ will be endowed with a scheme structure, things will change: for instance, as we shall see (cf. Example 3.1.66), it is not true that the only morphism $\operatorname{Spec} \mathbb{F} \to \operatorname{Spec} \mathbb{F}$ is the identity! And it is also not true that there exists a morphism $\operatorname{Spec} \mathbb{F} \to \operatorname{Spec} \mathbb{F}'$ for any pair of fields $\mathbb{F}$ and $\mathbb{F}'$.

Figure 3.1: This is $\operatorname{Spec} \mathbb{F}$. Nothing more, nothing less.

Example 3.1.21 ($\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$)**.** Let $\mathbf{k}$ be an algebraically closed field, such as $\mathbb{C}$. The ring $\mathbf{k}[x]$ is a principal ideal domain, whose prime ideals are (0) and $(x - a)$, one for each $a \in \mathbf{k}$. The spectrum

$$\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$$

is called the *affine line* (over $\mathbf{k}$). Note that there is exactly one point, namely

$$\xi = (0) \in \mathbb{A}^1_{\mathbf{k}},$$

that is not closed. In fact, by Lemma 3.1.16, we have

$$\overline{\{\xi\}} = \mathrm{V}(0) = \mathbb{A}^1_{\mathbf{k}}.$$

This point was invisible in the land of *classical varieties*, where only closed points were allowed. It has a name: it is the *generic point* of the affine line. We will say a lot more about generic points later (cf. Section 3.1.6), but for now notice that the terminology is somewhat well chosen: if we think that $(x-a)$ corresponds to the 'classical' point $a \in \mathbb{C}$, then since $x - x = 0$ it is reasonable to think that the coordinate of this point has indeed stayed 'generic'. This is what an 'indeterminate' should be!

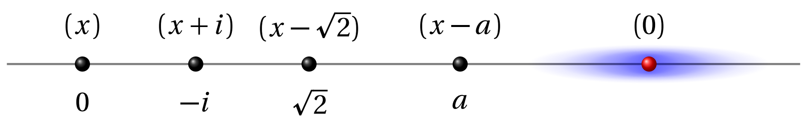


Figure 3.2: The topological space $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[x]$, with one closed point for every $a \in \mathbb{C}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \mathbb{A}^1_{\mathbb{C}}$, since $0 \in \mathbb{C}[x]$ is in every prime ideal.

Note that, if $\mathbb{F}$ is an arbitrary field, not necessarily algebraically closed, $\mathbb{F}[x]$ is still a principal ideal domain, but now the closed points of $\mathbb{A}^1_{\mathbb{F}}$ correspond to those maximal ideals (f) generated by irreducible polynomials of degree possibly larger than 1. The composition

$$\mathbb{F} \hookrightarrow \mathbb{F}[x] \twoheadrightarrow \mathbb{F}[x]/(f)$$

is a finite extension of fields, of degree equal to $\deg f$ (cf. Example 3.1.23).

Example 3.1.22 (Spec $\mathbb{Z}$)**.** The spectrum $\operatorname{Spec}\mathbb{Z}$ is the arithmetic counterpart of $\operatorname{Spec}\mathbf{k}[x]$. It has one closed point for every nonzero prime ideal $(p) \subset \mathbb{Z}$, and, again, precisely one non-closed point $\xi = (0) \in \operatorname{Spec}\mathbb{Z}$ called the generic point.

Figure 3.3: The topological space $\operatorname{Spec}\mathbb{Z}$, with one closed point for every prime $p \in \mathbb{Z}$. The generic point $\xi = (0)$ is 'dense', i.e. $\overline{\{\xi\}} = \operatorname{Spec}\mathbb{Z}$, since $0 \in \mathbb{Z}$ is in every prime ideal.

Example 3.1.23 ($\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[x]$)**.** The ring $\mathbb{R}[x]$ is a principal ideal domain. Its prime ideals are

$$(0), \quad (x-a), \quad (x^2+bx+c),$$

where $a \in \mathbb{R}$ and x^2+bx+c is irreducible and satisfies $b^2-4c<0$. The only prime ideal which is not maximal is, once more, $(0) \subset \mathbb{R}[x]$. However, we see here an important phenomenon arising when one considers fields that are not algebraically closed: we have, for the two types of *maximal* ideals,

$$\mathbb{R}[x]/(x-a) \cong \mathbb{R}, \quad \mathbb{R}[x]/(x^2+bx+c) \cong \mathbb{C}.$$

Of course, if f is an irreducible quadratic polynomial as above, the prime ideal $(f) \subset \mathbb{R}[x]$ defines one precise point in the spectrum, although we may want to think of it as the identification of two complex conjugate points, just as $\pm i$ give rise to $x^2+1=(x-i)(x+i)$. See Example 3.1.81 for more on this.

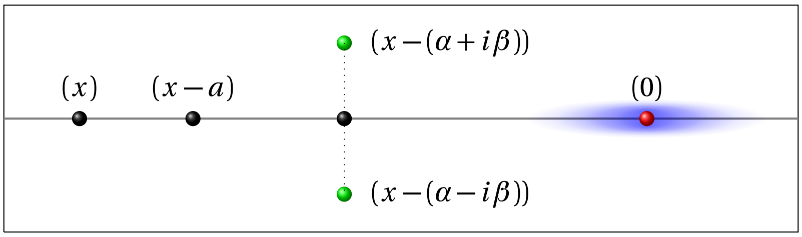


Figure 3.4: The real affine line $\mathbb{A}^1_{\mathbb{R}}$. Points $(f)\in\mathbb{A}^1_{\mathbb{R}}$ with $\mathbb{R}[x]/(f)\cong\mathbb{C}$ can be thought of pairs of conjugate *complex* points coming together.

Example 3.1.24 ($\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$)**.** Let $\mathbf{k}$ be an algebraically closed field. The spectrum $\mathbb{A}^2_{\mathbf{k}}=\operatorname{Spec}\mathbf{k}[x,y]$ is called the *affine plane* (over $\mathbf{k}$). The prime ideals in $\mathbf{k}[x,y]$ are

$$(0),\quad (x-a,y-b),\quad (f)$$

where $(a,b)\in\mathbf{k}^2$ and $f=f(x,y)$ is an irreducible polynomial. Maximal ideals are those of the form $(x-a,y-b)$, and correspond indeed to the 'classical' points (a,b) of $\mathbf{k}^2$. These are then closed points of $\mathbb{A}^2_{\mathbf{k}}$. Given an irreducible polynomial $f\in\mathbf{k}[x,y]$, we have

$$\mathrm{V}(f)=\left\{\mathfrak{p}\in\mathbb{A}^2_{\mathbf{k}}\,\middle|\,f\in\mathfrak{p}\right\}=\left\{(x-a,y-b)\,\middle|\,f(a,b)=0\right\}\cup\left\{(f)\right\}.$$

Clearly $\mathrm{V}(f)$ is the closure of $\{(f)\}$. The ideal (f) is the generic point of $\mathrm{V}(f)$, because it corresponds to the trivial ideal in the integral domain $\mathbf{k}[x,y]/(f)$, whereas the other points are closed points.

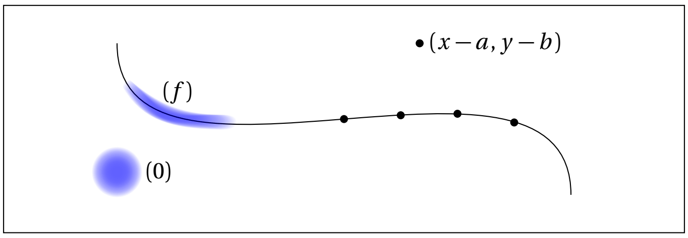


Figure 3.5: The affine plane $\mathbb{A}^2_{\mathbf{k}}$.

Example 3.1.25 (Spec of a DVR)**.** Let A be a DVR, shorthand for 'discrete valuation ring'. Then, by definition, A is a principal ideal domain with exactly one maximal ideal $\mathfrak{m}\subset A$. This ideal is also prime, and there is precisely one other prime ideal, namely $(0)\subset A$. In other words,

$$\operatorname{Spec}A=\{\xi,\mathfrak{m}\}$$

consists of two points, where $\mathfrak{m}$ is closed (cf. Remark 3.1.15) and hence ξ, corresponding to $(0)\subset A$, is open. Note that the Zariski topology is *not the discrete topology* on two points here, for the point $\mathfrak{m}$ is a *specialisation* of the point ξ, i.e. $\mathfrak{m}$ lies in the closure of ξ. We shall see later that, despite being a finite set, $\operatorname{Spec} A$ is a 1-dimensional scheme (or A is a 1-dimensional ring, cf. **??**), simply because $0 \in \mathfrak{m}$. An example of DVR is given by the ring of formal power series $\mathbf{k}[\![t]\!]$, where $\mathbf{k}$ is a field. In this case, the maximal ideal is just the ideal generated by t.

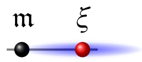


Figure 3.6: The black bullet represents the closed point $\mathfrak{m}$. The red point surrounded by the cloud, as usual, is the generic point.

Example 3.1.26 ($\operatorname{Spec} \mathbf{k}[t]/t^2$)**.** First of all some terminology: the ring $A = \mathbf{k}[t]/t^2$ is called the *ring of dual numbers*[2] (over $\mathbf{k}$). Some people write $\mathbf{k}[\varepsilon]$ to denote this ring, being understood that $\varepsilon^2 = 0$. There is only one prime (and in fact maximal) ideal in A, namely

$$(\overline{t}) \subset A,$$

where $\overline{t}$ is the image of $t \in \mathbf{k}[t]$ under the projection $\mathbf{k}[t] \twoheadrightarrow A$. Thus, topologically, this space is the same as $\operatorname{Spec} \mathbf{k}$. However, it will be different (i.e. *not isomorphic* to $\operatorname{Spec} \mathbf{k}$) as an affine scheme. A first strong indication of this fact was already given in Example 3.1.19.

3.1.4 The sheaf of rings $\mathscr{O}_{\operatorname{Spec} A}$

Let A be a ring. Set $X = \operatorname{Spec} A$, equipped as always with the Zariski topology. We now define a sheaf of rings

$$\mathscr{O}_X \in \mathsf{Sh}(X, \mathsf{Rings}),$$

that we will refer to as the *sheaf of regular functions* on $X = \operatorname{Spec} A$.

By Lemma 2.7.7, to define a sheaf of rings on the topological space X, it is enough to define a $\mathscr{B}$-sheaf of rings where

$$\mathscr{B} = \left\{ \mathrm{D}(f) \,\middle|\, f \in A \right\} \subset \tau_X$$

is the base of principal open sets in X (cf. Corollary 3.1.7). Our working definition for this $\mathscr{B}$-sheaf will be the assignment

$$\mathrm{D}(f) \longmapsto A_f = \left\{ \frac{a}{f^n} \,\middle|\, a \in A, n \geq 0 \right\}. \tag{3.1.4}$$

[2]In case you care to know why they have this name, here is the answer directly from Wikipedia: Dual numbers were introduced in 1873 by William Clifford, and were used at the beginning of the twentieth century by the German mathematician Eduard Study, who used them to represent the dual angle which measures the relative position of two skew lines in space.

Note that (if we take $f = 1$) we are *defining*

$$\mathscr{O}_X(X) = A.$$

See Appendix B.4 for all you need to know about localisation. Sometimes we shall write af^{-n} or a/f^n for the element

$$\frac{a}{f^n} \in A_f.$$

We need to verify that (3.1.4) does indeed define a $\mathcal{B}$-sheaf of rings.

First of all, let us make sure this assignment is well-defined. We know (cf. Example 3.1.11) that

$$\mathrm{D}(g) \subset \mathrm{D}(f) \iff g \in \sqrt{fA} \iff \frac{f}{1} \in A_g^\times.$$

For the second equivalence, write $g^r = fb$ in A for some $b \in A$ and some $r > 0$. Thus

$$\frac{f}{1} \cdot \frac{b}{1} = \frac{fb}{1} = \frac{g^r}{1} \in A_g,$$

which is invertible in A_g. Therefore

$$\frac{f}{1} \in A_g$$

is also invertible, with inverse

$$\left(\frac{f}{1}\right)^{-1} = \frac{1}{g^r} \cdot \frac{b}{1} = \frac{b}{g^r} \in A_g.$$

By the universal property of the localisation A_f, we get a canonical ring homomorphism

(3.1.5)
$$\begin{array}{ccc} A_f & \xrightarrow{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} & A_g \\ \dfrac{a}{f^n} & \longmapsto & \dfrac{ab^n}{g^{nr}} \end{array}$$

making the diagram

$$\begin{array}{ccc} A & \longrightarrow & A_g \\ \downarrow & \nearrow_{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} & \\ A_f & & \end{array}$$

commute. This map is an isomorphism as soon as $\mathrm{D}(g) = \mathrm{D}(f)$, showing that (3.1.4) is well-defined.

Note that the assignment (3.1.4) prescribes (cf. Remark B.4.5)

$$\emptyset = \mathrm{D}(0) \mapsto A_0 = 0,$$

The following lemma confirms that the maps just defined compose well, thus turning $\mathrm{D}(f) \mapsto A_f$ into a $\mathcal{B}$-presheaf.

LEMMA 3.1.27. *Fix* $f, g, h \in A$.

(i) *We have* $\rho_{\mathrm{D}(f)\mathrm{D}(f)} = \mathrm{id}_{A_f}$.

(ii) *Given inclusions of principal open subsets*

$$\mathrm{D}(h) \subset \mathrm{D}(g) \subset \mathrm{D}(f)$$

in $\operatorname{Spec} A$*, we have an identity*

$$\rho_{\mathrm{D}(g)\mathrm{D}(h)} \circ \rho_{\mathrm{D}(f)\mathrm{D}(g)} = \rho_{\mathrm{D}(f)\mathrm{D}(h)}$$

as maps $A_f \to A_h$.

In particular, $\mathrm{D}(f) \mapsto A_f$ *defines a* $\mathcal{B}$*-presheaf on* $\operatorname{Spec} A$.

Proof. Condition (i) is clear, so we move to (ii). First we write

$$g^r = f b, \quad h^s = g c,$$

for some $r, s > 0$ and $b, c \in A$. Then

$$h^{rs} = (h^s)^r = g^r c^r = f b c^r.$$

Then, according to (3.1.5), the map $\rho_{\mathrm{D}(f)\mathrm{D}(h)} \colon A_f \to A_h$ is given by

$$\frac{a}{f^n} \mapsto \frac{a(bc^r)^n}{h^{rsn}}. \tag{3.1.6}$$

On the other hand, we have to compose

$$\begin{array}{ccc} A_f & \xrightarrow{\rho_{\mathrm{D}(f)\mathrm{D}(g)}} & A_g \\ \dfrac{a}{f^n} & \longmapsto & \dfrac{ab^n}{g^{nr}} \end{array}$$

and

$$\begin{array}{ccc} A_g & \xrightarrow{\rho_{\mathrm{D}(g)\mathrm{D}(h)}} & A_h \\ \dfrac{a}{g^m} & \longmapsto & \dfrac{ac^m}{h^{ms}} \end{array}$$

with one another. The result of the composition is

$$\frac{a}{f^n} \mapsto \frac{ab^n}{g^{nr}} \mapsto \frac{ab^n c^{nr}}{h^{nrs}} = \frac{a(bc^r)^n}{h^{rsn}},$$

which agrees with (3.1.6), as we wanted. □

THEOREM 3.1.28. *Let* A *be a ring. Set* $X = \operatorname{Spec} A$.

(a) *The rule* (3.1.4) *defines a* $\mathcal{B}$*-sheaf of rings on* X*. The induced sheaf of rings will be denoted* $\mathcal{O}_X$*.*

(b) *We have* $\mathcal{O}_X(X) = A$*.*

(c) *The stalk* $\mathcal{O}_{X,x}$ *of* $\mathcal{O}_X$ *at the point* $x \in X$ *corresponding to* $\mathfrak{p} \subset A$ *is isomorphic to* $A_\mathfrak{p}$*.*

Proof. We proceed step by step.

(a) We know that (3.1.4) defines a $\mathcal{B}$-presheaf of rings by Lemma 3.1.27. We check the sheaf conditions (3)–(4) of Important Definition 2.2.1 on the open set $U = X = \mathrm{D}(1)$, the case of an arbitrary principal open $U = \mathrm{D}(h) \in \mathcal{B}$ being essentially identical. Recall from (3.1.3) that

$$\operatorname{Spec} A = \bigcup_{f \in A} \mathrm{D}(f).$$

By Lemma 3.1.12, this is equivalent to saying that there is a *finite* set F indexing a set of generators $\{ f_i \mid i \in F \} \subset A$ of the unit ideal $(1) = A$, so that in particular $1 \in \sum_{i \in F}(f_i)$. In what follows, set $U_i = \mathrm{D}(f_i)$ and $U_{ij} = U_i \cap U_j = \mathrm{D}(f_i f_j)$.

Sheaf axiom (3): Fix $s \in A_1 = A$ such that $s|_{U_i} = 0 \in A_{f_i}$. This means

$$\frac{s}{1} = \frac{0}{1} \in A_{f_i},$$

i.e. there exists $m > 0$ such that $f_i^m s = 0 \in A$. Since F is finite, we can pick a uniform m which works for every f_i. Since

$$X = \bigcup_{i \in F} \mathrm{D}(f_i) = \bigcup_{i \in F} \mathrm{D}(f_i^m),$$

as before we have

$$1 \in \sum_{i \in F} (f_i^m),$$

which implies

$$s \in \sum_{i \in F} (f_i^m s) = 0.$$

Hence $s = 0$, as required.

Sheaf axiom (4): By definition, $\mathcal{O}_X(U_{ij}) = A_{f_i f_j}$. Fix sections $s_i \in A_{f_i}$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for every i and j. That is, s_i and s_j have the same image along the maps

$$A_{f_i} \longrightarrow A_{f_i f_j} \longleftarrow A_{f_j}$$

Write (again for a uniform $m > 0$)

$$s_i = \frac{b_i}{f_i^m} \in A_{f_i}, \quad i \in F.$$

Now, $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ means that there exists an integer $r > 0$ such that

$$(f_i f_j)^r (b_i f_j^m - b_j f_i^m) = 0 \in A \tag{3.1.7}$$

for all $i, j \in F$. As before,

$$1 \in \sum_{i \in F} (f_i^{m+r})$$

yields

$$1 = \sum_{j \in F} a_j f_j^{m+r}, \quad a_j \in A. \tag{3.1.8}$$

Define

$$s = \sum_{j \in F} a_j b_j f_j^r \in A, \tag{3.1.9}$$

so that the chain of identities

$$\begin{aligned} f_i^{m+r} s &= \sum_{j \in F} a_j b_j f_i^m (f_i f_j)^r && \text{by (3.1.9)} \\ &= \sum_{j \in F} a_j b_i f_j^m (f_i f_j)^r && \text{by (3.1.7)} \\ &= \sum_{j \in F} a_j f_j^{m+r} b_i f_i^r \\ &= b_i f_i^r && \text{by (3.1.8)} \end{aligned}$$

yields

$$f_i^r (b_i - f_i^m s) = 0.$$

But this in turn is equivalent to

$$\frac{s}{1} = \frac{b_i}{f_i^m} = s_i \in A_{f_i}.$$

So we have proved $s|_{U_i} = s_i$ for every $i \in F$.

(b) We have $\operatorname{Spec} A = \mathrm{D}(1)$, so this actually follows from the definition, using that $A_1 = A$ since $1 \in A$ is already invertible in A.

(c) Let $\mathfrak{p} \subset A$ be the ideal corresponding to $x \in X$. For every $f \notin \mathfrak{p}$, there is (by the universal property of A_f) a canonical map $A_f \to A_{\mathfrak{p}} = \{a/h \mid h \notin \mathfrak{p}\}$ because f is invertible in $A_{\mathfrak{p}}$. By the universal property of colimits, we get a canonical ring homomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{f \notin \mathfrak{p}} A_f \xrightarrow{\ \alpha\ } A_{\mathfrak{p}}.$$

An element of the form $a/h \in A_{\mathfrak{p}}$ lies in the image of $A_h \to \mathscr{O}_{X,x} \to A_{\mathfrak{p}}$, therefore α is surjective. On the other hand, if $a/h^n \in A_h$ (for $h \notin \mathfrak{p}$ and some $n > 0$) maps to $0 = 0/1 \in A_{\mathfrak{p}}$, then by definition of localisation there exists $g \in A \setminus \mathfrak{p}$ such that $ga = 0 \in A$. Then the image of a/h^n in A_{gh} is

$$\frac{g^{n-1}ga}{(gh)^n} = 0 \in A_{gh},$$

so a/h^n goes to 0 in $\mathscr{O}_{X,x}$. We have confirmed that

$$\ker\big(A_h \to \mathscr{O}_{X,x} \to A_{\mathfrak{p}}\big) = \ker\big(A_h \to \mathscr{O}_{X,x}\big) \tag{3.1.10}$$

for every $h \in A \setminus \mathfrak{p}$, which is enough to conclude that α is injective. (Reason: Let $z \in \mathscr{O}_{X,x}$ be an element such that $\alpha(z) = 0 \in A_{\mathfrak{p}}$. There exist $a \in A$, $h \in A \setminus \mathfrak{p}$ and $n \geq 0$ such that $z = q(a/h^n)$ where $q\colon A_h \to \mathscr{O}_{X,x}$ is the canonical map. So $0 = \alpha(q(a/h^n))$ implies $a/h^n \in \ker(\alpha \circ q) = \ker q$, the identity of kernels being our assumption (3.1.10). It follows that $0 = q(a/h^n) = z$, as required). □

The following is now immediate from the definition of locally ringed space.

COROLLARY 3.1.29. *Let A be a ring. The pair* $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ *defines a locally ringed space. For every $\mathfrak{p} \in \operatorname{Spec} A$, the corresponding local ring at $\mathfrak{p}$ is the local ring* $(A_{\mathfrak{p}}, \mathfrak{p}A_{\mathfrak{p}})$.

The quotient $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is called the *residue field* at $\mathfrak{p}$.

Important Definition 3.1.2 (Affine scheme)**.** An *affine scheme* is a locally ringed space isomorphic (in the category of locally ringed spaces) to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A.

As an important class of examples of affine schemes, we have the notion of affine algebraic variety. We will see a different notion, that of projective variety, in Important Definition 3.2.1.

Important Definition 3.1.3 (Affine variety)**.** An *affine variety* over a field $\mathbb{F}$ (also called an *affine $\mathbb{F}$-variety*) is an affine scheme of the form $\operatorname{Spec} A$, where A is a finitely generated $\mathbb{F}$-algebra (i.e. $A = \mathbb{F}[x_1, \ldots, x_n]/I$ for some n and some ideal I).

Exercise 3.1.30. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Set $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Show that there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ A/\mathfrak{p} & \hookrightarrow & \kappa(\mathfrak{p}) \end{array}$$

of rings, and that

$$\operatorname{Frac} A/\mathfrak{p} = \kappa(\mathfrak{p}).$$

In particular, if $\mathfrak{m} \subset A$ is maximal, then $\kappa(\mathfrak{m}) = A/\mathfrak{m}$.

Example 3.1.31. Let $\mathbb{F}$ be a field, $0 \in X = \mathbb{A}^1_{\mathbb{F}} = \operatorname{Spec}\mathbb{F}[x]$ the point corresponding to $(x) \subset \mathbb{F}[x]$. (You are allowed, and in fact encouraged, to call this point 'the origin' of the affine line). The local ring of X at 0 is

$$\mathscr{O}_{X,0} = \mathbb{F}[x]_{(x)} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(x) \notin (x) \right\} = \left\{ \frac{f(x)}{g(x)} \,\middle|\, g(0) \neq 0 \right\} \subset \operatorname{Frac}\mathbb{F}[x] = \mathbb{F}(x),$$

and the residue field is

$$\kappa(0) = \mathscr{O}_{X,0}/\mathfrak{m}_0 = \mathbb{F}[x]_{(x)}/(x)\mathbb{F}[x]_{(x)} \cong \mathbb{F}[x]/(x) \cong \mathbb{F}.$$

For the second-last isomorphism we used Exercise 3.1.30 (see also Proposition B.4.10). The same chain of isomorphisms holds replacing 0 with any other closed point of the form $(x-a)$, with $a \in \mathbb{F}$. If $\xi = (0)$, we have

$$\kappa(\xi) = \mathscr{O}_{X,\xi} = \mathbb{F}(x).$$

Functions on $\operatorname{Spec} A$ revisited

We already saw, but we need to emphasise, that

$$\boxed{\mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A.}$$

That is, regular functions on $\operatorname{Spec} A$ are precisely the elements of A.

We could have bypassed $\mathscr{B}$-sheaves and defined the sheaf of rings $\mathscr{O}_X$ on $X = \operatorname{Spec} A$ directly (as done in [8, Ch. 2]) by setting

$$\mathscr{O}_X(U) = \left\{ U \xrightarrow{s} \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \,\middle|\, \begin{array}{c} \text{for every } \mathfrak{p} \in U,\ s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and there exist} \\ \text{an open neighbourhood } V \subset U \text{ of } \mathfrak{p} \\ \text{and } a, f \in A \text{ such that, for every } \mathfrak{q} \in V, \\ f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \text{ in } A_{\mathfrak{q}} \end{array} \right\} \tag{3.1.11}$$

for every open subset $U \subset X$. The fact that $U \mapsto \mathscr{O}_X(U)$ is a sheaf (and coincides with the sheaf $\mathscr{O}_X$ defined in Theorem 3.1.28(a)) is clear once one realises that the very definition just rephrases the notion of compatible germs.

Let us focus on the case $U = X = \operatorname{Spec} A$. Consider the map $\psi\colon A \to \mathscr{O}_X(X)$ sending $a \in A$ to the function

$$s_a\colon X \to \coprod_{\mathfrak{p}\in X} A_{\mathfrak{p}}, \quad \mathfrak{p} \mapsto \text{image of } a \text{ along } A \to A_{\mathfrak{p}}.$$

This map ψ is injective. Indeed, assume $s_a = s_b$ for $a, b \in A$. This means that for every $\mathfrak{p} \in X$, the elements a and b have the same image in $A_{\mathfrak{p}}$. Hence there is an element $r \in A \setminus \mathfrak{p}$ such that $r(a-b) = 0$ in A. Set $J = \operatorname{Ann}(a-b)$, so that $r \in J$. Thus $J \not\subset \mathfrak{p}$ for every $\mathfrak{p}$. But then, in particular, J is not contained in any maximal ideal, hence $J = A$. Thus $a - b = 1 \cdot (a-b) = 0$, i.e. ψ is injective.

One may insist to call *regular function* on $X = \operatorname{Spec} A$ a map that is *field-valued*. This can be done as follows, starting from the definition in Equation (3.1.11). Let $a \in A = \mathscr{O}_X(X)$. Composing s_a with the quotient maps

$$A_{\mathfrak{p}} \twoheadrightarrow \kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$$

one obtains the map

$$\widetilde{s}_a : X \longrightarrow \coprod_{\mathfrak{p}\in X} \kappa(\mathfrak{p}),$$

where the field $\kappa(\mathfrak{p})$ may (and will) vary from point to point (cf. Example 3.1.54).

Remark 3.1.32. The closed set $\mathrm{V}(I) = \{\,\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset I\,\} \subset \operatorname{Spec} A$ can be reinterpreted as

$$\mathrm{V}(I) = \{\,\mathfrak{p} \in \operatorname{Spec} A \mid \widetilde{s}_a(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p}) \text{ for all } a \in I\,\}. \tag{3.1.12}$$

This explains once more the letter 'V', standing for 'vanishing'. But if you encounter the letter 'Z', it stands for 'zero locus'!

3.1.5 The definition of schemes and first topological properties

We are ready for the definition of schemes.

Important Definition 3.1.4 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point $x \in X$ has an open neighbourhood $x \in U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

Terminology 3.1.33. Let $(X, \mathscr{O}_X)$ be a scheme. The structure sheaf $\mathscr{O}_X$ is referred to as the *sheaf of regular functions* of the scheme. The ring $\mathscr{O}_X(X)$ is called the ring of *regular functions* on X. We keep the notation $(\mathscr{O}_{X,x}, \mathfrak{m}_x, \kappa(x))$ for the local ring at a point $x \in X$.

Definition 3.1.34 (Morphism of schemes)**.** A *morphism of schemes* is a morphism in the category of locally ringed spaces. In particular, an *isomorphism of schemes* $(X, \mathscr{O}_X) \xrightarrow{\sim} (Y, \mathscr{O}_Y)$ is a morphism $(f, f^\#)$ such that $f : X \xrightarrow{\sim} Y$ is a homeomorphism and $f^\# : \mathscr{O}_Y \xrightarrow{\sim} f_*\mathscr{O}_X$ is an isomorphism of sheaves of rings over Y.

Definition 3.1.35 (Immersions of schemes)**.** An open (resp. closed) immersion of schemes is an open (resp. closed) immersion in the category of locally ringed spaces (cf. Definition 2.10.8).

Definition 3.1.36 (Open subscheme)**.** An *open subscheme* of a scheme $(X, \mathscr{O}_X)$ is a scheme of the form $(U, \mathscr{O}_X|_U)$, where U is an open subset of X. We will often just write $\mathscr{O}_U$ instead of $\mathscr{O}_X|_U$.

Note that an open subscheme comes with an open immersion $U \hookrightarrow X$. We will see in Remark 3.1.59 that an open subset $U \subset X$ of a scheme X is naturally a scheme.

The definition of closed subscheme is more subtle.

Definition 3.1.37 (Closed subscheme)**.** Let X be a scheme. A *closed subscheme* of X is an equivalence class of closed immersions with target X, where two closed immersions $\iota\colon Z \hookrightarrow X$ and $\iota'\colon Z' \hookrightarrow X$ are isomorphic if there exists an isomorphism $\alpha\colon Z \xrightarrow{\sim} Z'$ such that $\iota' \circ \alpha = \iota$.

Notation 3.1.38. Affine schemes (resp. schemes) form a category, denoted Aff (resp. Sch), where morphisms are just the morphisms in the larger category of locally ringed spaces. We denote a scheme $(X, \mathscr{O}_X)$ simply by X, and a morphism $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ simply by $f\colon X \to Y$.

We have thus a chain of full inclusions of categories

$$\mathsf{Aff} \hookrightarrow \mathsf{Sch} \hookrightarrow \mathsf{LRS}.$$

Here are some purely topological properties of a scheme. Recall that a topological space X is *irreducible* if it cannot be written as a union $Z_1 \cup Z_2$ of two proper closed subsets $Z_i \subset X$. Equivalently, X is irreducible if and only if any two nonempty open subsets of X intersect. Moreover, any open subset U of an irreducible topological space is dense, i.e. $\overline{U} = X$.

Definition 3.1.39. A scheme $(X, \mathscr{O}_X)$ is said to be *quasicompact* (resp. *irreducible*, resp. *connected*) if the underlying topological space X is. A morphism of schemes $f\colon X \to Y$ is called *quasicompact* if the preimage of any affine open subset is quasicompact.

Translation: a scheme is quasicompact when it admits a finite open cover by affine schemes. For morphism, we have the following.

Exercise 3.1.40. A morphism $f\colon X \to Y$ is quasicompact if and only if Y has an affine open cover $Y = \bigcup_{i \in I} Y_i$ such that $f^{-1}(Y_i)$ is quasicompact for all i.

We already saw in Lemma 3.1.12 than an affine scheme is quasicompact. Any irreducible scheme is in particular connected. There are, however, connected schemes which are reducible (i.e. not irreducible), see Example 3.1.44.

PROPOSITION 3.1.41. *If X is a quasicompact scheme, then X has a closed point.*

Proof. We present Schwede's proof [14, Prop. 4.1].

By quasicompactness of X, there is a finite open cover of X by affine schemes $U_i = \operatorname{Spec} A_i$, say $X = U_1 \cup \cdots \cup U_r$. Consider a closed point $x_1 \in U_1$. If x_1 is closed in X, we are done. Otherwise, pick a point $x_2 \in \overline{\{x_1\}}$, with $x_2 \neq x_1$. Then x_2 lies in some U_i, but not in

U_1 since $\overline{\{x_1\}} \cap U_1 = \{x_1\}$. Say $x_2 \in U_2 \setminus U_1$. If x_2 is closed in X, we are done. Otherwise, pick a point $x_3 \in \overline{\{x_2\}}$, with $x_3 \neq x_2$. But x_3 is also in the closure of x_1, thus $x_3 \notin U_1 \cup U_2$. Say $x_3 \in U_3$, and continue until the cover is exhausted: this process stops, so X must contain a closed point. □

Exercise 3.1.42. Prove that a scheme X is the spectrum of a local ring if and only if it is quasicompact and has a unique closed point.

3.1.6 Generic points, take I

We start here our discussion around generic points. We cover, for now, only the case of irreducible schemes. You will notice that most arguments are entirely of topological nature.

The next result tells us when an affine scheme, or a closed subset thereof, is irreducible.

PROPOSITION 3.1.43. *Let A be a ring, and set $X = \operatorname{Spec} A$. Then a closed subset* $\mathrm{V}(I) \subset X$ *is irreducible if and only if $\sqrt{I} \subset A$ is prime. In particular,*

(a) *X is irreducible if and only if $\sqrt{0} \subset A$ is prime.*

(b) *If A is an integral domain, then* $\operatorname{Spec} A$ *is irreducible.*

Proof. Assume $\mathrm{V}(I) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset I\} \subset X$ irreducible. To say that $\sqrt{I}$ is prime means that if $ab \in \sqrt{I}$ then either $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Let us assume, by contradiction, that there are $a, b \in A \setminus \sqrt{I}$ such that $ab \in \sqrt{I}$. Set $X_a = \mathrm{V}(I) \cap \mathrm{V}(a)$ and $X_b = \mathrm{V}(I) \cap \mathrm{V}(b)$. Then $X_a \cup X_b \subset \mathrm{V}(I)$, and if $\mathfrak{p} \in \mathrm{V}(I) = \mathrm{V}(\sqrt{I})$ then $\mathfrak{p} \supset \sqrt{I} \ni ab$, so that either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, which proves $X_a \cup X_b = \mathrm{V}(I)$. Since $X_a \neq \mathrm{V}(I) \neq X_b$, we contradict irreducibility of $\mathrm{V}(I)$.

Conversely, assume $\sqrt{I} \subset A$ is prime. If $\mathrm{V}(I) = \mathrm{V}(J_1) \cup \mathrm{V}(J_2) = \mathrm{V}(J_1 \cap J_2) = \mathrm{V}(J_1 J_2)$ then $\sqrt{I} = \sqrt{J_1 J_2} \supset J_1 J_2$, thus either $J_1 \subset \sqrt{I}$ or $J_2 \subset \sqrt{I}$. But if, say, $J_1 \subset \sqrt{I}$, it follows that $\mathrm{V}(I) = \mathrm{V}(\sqrt{I}) \subset \mathrm{V}(J_1)$, which yields $\mathrm{V}(J_1) = \mathrm{V}(I)$. Thus $\mathrm{V}(I)$ is irreducible. □

Example 3.1.44. The closed subset

$$\mathrm{V}(y - x^2) \subset \mathbb{A}^2_{\mathbf{k}}$$

is irreducible. On the other hand, the closed subset

$$\mathrm{V}(y^2 - x^2) \subset \mathbb{A}^2_{\mathbf{k}}$$

is reducible (but connected), being equal to $\mathrm{V}(x - y) \cup \mathrm{V}(x + y)$. The same conclusion holds for $\mathrm{V}(xy) \subset \mathbb{A}^2_{\mathbf{k}}$.

Caution 3.1.45 (Irreducibility depends on the base field)**.** Consider the polynomial $f = x^2+1 \in \mathbb{R}[x] \subset \mathbb{C}[x]$. Then $\mathrm{V}(f) \subset \mathbb{A}^1_{\mathbb{R}}$ is irreducible (a point), but $\mathrm{V}(f) \subset \mathbb{A}^1_{\mathbb{C}}$ is reducible, being equal to $\mathrm{V}(x-i) \cup \mathrm{V}(x+i)$.

LEMMA 3.1.46. *Let X be an irreducible scheme. Then, there exists a unique point $\xi \in X$ such that $X = \overline{\{\xi\}}$.*

Proof. Let us show uniqueness first. Let ξ_1, ξ_2 be two points such that $\overline{\{\xi_1\}} = X = \overline{\{\xi_2\}}$. Then any nonempty open $U \subset X$ contains both ξ_1 and ξ_2. Pick a nonempty affine open subset $U = \operatorname{Spec} A \subset X$. Let $\mathfrak{p}_i \subset A$ be the prime ideal corresponding to ξ_i. Since X is irreducible, U is irreducible and dense, hence

$$U = \overline{\{\xi_i\}} = \mathrm{V}(\mathfrak{p}_i), \quad i = 1, 2,$$

where the closure is taken in U. So when we write $\xi_2 \in U = \mathrm{V}(\mathfrak{p}_1)$ we obtain $\mathfrak{p}_2 \supset \mathfrak{p}_1$, and when we write $\xi_1 \in U = \mathrm{V}(\mathfrak{p}_2)$ we obtain $\mathfrak{p}_1 \supset \mathfrak{p}_2$. Thus $\mathfrak{p}_1 = \mathfrak{p}_2$, i.e. $\xi_1 = \xi_2$.[3]

Now for existence. If $U = \operatorname{Spec} A \subset X$ is a nonempty open affine subset, then U is irreducible, i.e. $\mathfrak{p} = \sqrt{0} \subset A$ is prime by Proposition 3.1.43, and hence $U = \mathrm{V}(\sqrt{0}) = \mathrm{V}(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. But since U is dense in X, the closure of $\mathfrak{p}$ in X is X itself. □

Definition 3.1.47 (Generic point, take I)**.** Let X be an irreducible scheme. The point $\xi \in X$ of Lemma 3.1.46 is called the *generic point* of X.

Remark 3.1.48. The proof of Lemma 3.1.46 actually works for an arbitrary irreducible closed subset X of an arbitrary scheme Y.

Remark 3.1.49. If A is a domain, then $(0) \subset A$ is the unique minimal prime, thus $\operatorname{Spec} A$ is irreducible (Proposition 3.1.43). For instance, if $\mathbb{F}$ is a field, the affine space $\mathbb{A}^n_{\mathbb{F}}$ is irreducible. However, the converse is false: for instance, $\operatorname{Spec} \mathbb{F}[t]/t^n$ is irreducible for every $n \geq 1$, but $\mathbb{F}[t]/t^n$ is not a domain as soon as $n > 1$.

Exercise 3.1.50. Prove that a scheme X is connected if and only if $\mathscr{O}_X(X)$ has only the trivial idempotents $0, 1$.

Exercise 3.1.51. Let A, A' be rings, $\mathbb{F}$ a field. Decide whether the following affine schemes are irreducible (resp. connected):

(i) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^2(x+1))$,

(ii) $\operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3)$

(iii) $\operatorname{Spec} \mathbb{Z}[x]/(2x)$,

(iv) $\operatorname{Spec} \mathbb{C}[x,y]/(xy, y^2)$,

[3] Alternatively, just note that $\mathfrak{p}_1 = \sqrt{\mathfrak{p}_1} = \sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$.

(v) $\operatorname{Spec}\mathbb{F}[x,y]/(x^2,xy,y^3)$,

(vi) $\operatorname{Spec}(A\times A')$,

(vii) $\operatorname{Spec}\mathbb{C}[x,y,z]/(xy-z^2)$,

(viii) $\operatorname{Spec}\mathbb{C}[x,y]/(x^2+y^2-1)$.

It is clear from the last paragraph of the proof of Lemma 3.1.46 that if A is a domain then the generic point $\xi\in\operatorname{Spec}A$ is the point corresponding to $(0)\subset A$, which is manifestly the unique minimal prime. The following lemma clarifies the basic properties of the generic point in this special case.

LEMMA 3.1.52. *Let A be an integral domain with fraction field K. Let $\xi\in X=\operatorname{Spec}A$ be the point corresponding to $(0)\subset A$. Then*

(i) *We have $\mathscr{O}_{X,\xi}=K$.*

(ii) *ξ belongs to every nonempty open subset $U\subset X$, and $\mathscr{O}_X(U)\to\mathscr{O}_{X,\xi}$ is injective.*

(iii) *For every nonempty open subset $V\hookrightarrow U$, the map $\mathscr{O}_X(U)\to\mathscr{O}_X(V)$ is injective.*

Proof. We proceed step by step.

(i) This follows from Theorem 3.1.28(c) and the observation that the localisation of an integral domain at the prime ideal (0) is precisely the fraction field of the domain.

(ii) To say $\xi\in\operatorname{Spec}A\setminus\mathrm{V}(I)=\{\mathfrak{p}\in\operatorname{Spec}A\mid\mathfrak{p}\not\supset I\}$ for every $(0)\not\subset I\subset A$ means precisely that $(0)\not\supset I$ for all I, a tautology. Write $U=\bigcup_{i\in I}\mathrm{D}(f_i)$ and assume $s\in\mathscr{O}_X(U)$ goes to 0 in $K=\mathscr{O}_{X,\xi}$. Well, s goes to $s|_{\mathrm{D}(f_i)}\in\mathscr{O}_X(\mathrm{D}(f_i))=A_{f_i}$ first, and $A_{f_i}\hookrightarrow K$ is injective. Thus $s|_{\mathrm{D}(f_i)}=0$ for every $i\in I$, so $s=0$ by the sheaf conditions.

(iii) Follows immediately from (ii) and the factorisation $\mathscr{O}_X(U)\to\mathscr{O}_X(V)\to\mathscr{O}_{X,\xi}$. □

Example 3.1.53. Let $A=\mathbb{F}[x_1,\dots,x_n]$, and consider the generic point $\xi\in\mathbb{A}^n_{\mathbb{F}}$, i.e. the point corresponding to the ideal $(0)\subset A$. Then

$$\kappa(\xi)=\mathbb{F}(x_1,\dots,x_n).$$

If $f\in A\setminus 0$ is an irreducible polynomial, then the generic point $\xi_f\in\operatorname{Spec}A/(f)$ satisfies

$$\kappa(\xi_f)=\operatorname{Frac}A/(f).$$

Example 3.1.54. Let $A=\mathbb{Z}$. Every open subset $U\subset\operatorname{Spec}\mathbb{Z}$ is principal, i.e. of the form $U=\mathrm{D}(f)$ for some $f\in\mathbb{Z}$. We have $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f))=\mathbb{Z}_f\subset\operatorname{Frac}(\mathbb{Z})=\mathbb{Q}$, and a rational number $a/b\in\mathbb{Q}$ (with a,b coprime) belongs to $\mathscr{O}_{\operatorname{Spec}\mathbb{Z}}(\mathrm{D}(f))$ if and only if every prime

p dividing b also divides f. As for the generic point $\xi = (0)$, we have $\kappa(\xi) = \mathscr{O}_{\operatorname{Spec}\mathbb{Z},\xi} = \mathbb{Q}$. If $x \in \operatorname{Spec}\mathbb{Z}$ corresponds to the maximal ideal $(p) \subset \mathbb{Z}$, then

$$\kappa(x) = \mathbb{Z}_{(p)}/(p)\cdot\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$$

by Exercise 3.1.30. Therefore the residue fields at different points of a scheme can have different characteristic!

3.1.7 Morphisms of affine schemes

Let $\phi\colon A \to B$ be a ring homomorphisms. Then we have a set-theoretic map

$$f_\phi\colon \operatorname{Spec}B \to \operatorname{Spec}A, \quad \mathfrak{q} \mapsto \phi^{-1}(\mathfrak{q}).$$

LEMMA 3.1.55. *Let $\phi\colon A \to B$ be a ring homomorphisms. Then*

(a) *f_ϕ is continuous.*

(b) *If ϕ is surjective, then f_ϕ induces a homeomorphism from* $\operatorname{Spec}B$ *onto the closed subset* $\mathrm{V}(\ker\phi) \subset \operatorname{Spec}A$.

(c) *If ϕ is a localisation $A \to S^{-1}A$, then f_ϕ induces a homeomorphism from* $\operatorname{Spec}S^{-1}A$ *onto the subspace $Y_S = \{\,\mathfrak{p} \in \operatorname{Spec}A \mid \mathfrak{p}\cap S = \emptyset\,\} \subset \operatorname{Spec}A$.*

Proof. We proceed step by step.

(a) We prove that the preimage of a closed subset $\mathrm{V}(I) \subset \operatorname{Spec}A$ is closed. We have

$$\begin{aligned} f_\phi^{-1}(\mathrm{V}(I)) &= \{\,\mathfrak{q} \in \operatorname{Spec}B \mid \phi^{-1}(\mathfrak{q}) \in \mathrm{V}(I)\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec}B \mid I \subset \phi^{-1}(\mathfrak{q})\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec}B \mid \phi(I) \subset \mathfrak{q}\,\} \\ &= \{\,\mathfrak{q} \in \operatorname{Spec}B \mid IB \subset \mathfrak{q}\,\} \\ &= \mathrm{V}(IB). \end{aligned}$$

(b) We have $B = A/\ker\phi$ by assumption, and we know that the prime ideals of B are in bijection with the prime ideals of A containing $\ker\phi$. By (a), and by definition of $\mathrm{V}(-)$, we then know that f_ϕ factors through a continuous bijection $\operatorname{Spec}B \to \mathrm{V}(\ker\phi)$, still denoted f_ϕ. To conclude it is a homeomorphism, it is enough to check the map is closed. Let then $J \subset B$ be an ideal. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{\,\mathfrak{p} \in \operatorname{Spec}A \mid \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J\,\} \\ &= \{\,\mathfrak{p} \in \operatorname{Spec}A \mid \mathfrak{p} \supset \phi^{-1}(J)\,\} \\ &= \mathrm{V}(\phi^{-1}(J)). \end{aligned}$$

(c) The existence of a continuous bijection $\operatorname{Spec} S^{-1}A \to Y_S \subset \operatorname{Spec} A$ is a combination of (a) with Lemma B.4.6. As before, to see that the map is closed, fix an ideal $J \subset S^{-1}A$. Then

$$\begin{aligned} f_\phi(\mathrm{V}(J)) &= \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \emptyset \text{ and } \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \text{ for some } \mathfrak{q} \supset J \,\} \\ &= Y_S \cap \mathrm{V}(\phi^{-1}(J)), \end{aligned}$$

which is closed in Y_S, as required. □

Exercise 3.1.56. Let $\mathbb{F}$ be a field. Let $\phi\colon A \to B$ be a homomorphism of finitely generated $\mathbb{F}$-algebras. Show that $f_\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ maps closed points to closed points.

Remark 3.1.57. Note that if $S = \{\, g^m \mid m \geq 0 \,\}$ for some $g \in A$, then

$$Y_S = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid g \notin \mathfrak{p} \,\} = \mathrm{D}(g) \subset \operatorname{Spec} A.$$

In particular, in this case the map $f_\ell\colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ corresponding to the localisation $\ell\colon A \to A_g$ is a topological open embedding.

PROPOSITION 3.1.58. *Let $X = \operatorname{Spec} A$, and fix $g \in A$. The localisation $\ell\colon A \to A_g$ induces an isomorphism of locally ringed spaces*

$$(f_\ell, f_\ell^\#)\colon (\operatorname{Spec} A_g, \mathscr{O}_{\operatorname{Spec} A_g}) \xrightarrow{\ \sim\ } (\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)}).$$

In particular, $(\mathrm{D}(g), \mathscr{O}_X|_{\mathrm{D}(g)})$ *is an affine scheme.*

Proof. A topological open embedding $f_\ell\colon \operatorname{Spec} A_g \to \operatorname{Spec} A$ with image $\mathrm{D}(g)$ is provided by Lemma 3.1.55 (c), applied to the localisation $\ell\colon A \to A_g$. Let us denote by f the homeomorphism $\operatorname{Spec} A_g \to \mathrm{D}(g)$. We need to extend it to a morphism of locally ringed spaces and show the resulting map is an isomorphism.

To define a morphism of sheaves of rings

$$f^\#\colon \mathscr{O}_X|_{\mathrm{D}(g)} \to f_*\mathscr{O}_{\operatorname{Spec} A_g}$$

it is enough to define it on a base of open subsets by Proposition 2.7.9. Let $\mathrm{D}(h) \subset \mathrm{D}(g)$ be a principal open, for $h \in A$. Let $\overline{h} = h/1 \in A_g$ be the image of h in A_g. Then, canonically,

$$\begin{aligned} \mathscr{O}_X|_{\mathrm{D}(g)}(\mathrm{D}(h)) = \mathscr{O}_X(\mathrm{D}(h)) &= A_h \\ &\xrightarrow{\sim} (A_g)_{\overline{h}} \\ &= \mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(\overline{h})) \\ &= \mathscr{O}_{\operatorname{Spec} A_g}(f^{-1}\mathrm{D}(h)) \\ &= f_*\mathscr{O}_{\operatorname{Spec} A_g}(\mathrm{D}(h)). \end{aligned}$$

The isomorphism $A_h \xrightarrow{\sim} (A_g)_{\overline{h}}$ follows by directly checking that $(A_g)_{\overline{h}}$ satisfies the universal property of A_h. The above isomorphisms $\mathscr{O}_X|_{D(g)}(D(h)) \xrightarrow{\sim} f_* \mathscr{O}_{\operatorname{Spec} A_g}(D(h))$ are compatible with restrictions to smaller principal opens, thus they determine an isomorphism of $\mathcal{B}$-sheaves, which in turn uniquely determines an isomorphism of sheaves by Proposition 2.7.9. □

Remark 3.1.59. Let $(X, \mathscr{O}_X)$ be a scheme, $U \subset X$ an open subset. We know that $(U, \mathscr{O}_U)$ is a locally ringed space. (Recall that we set $\mathscr{O}_U = \mathscr{O}_X|_U$ in such a situation). We claim that it is in fact a scheme. That is, every open subset of a scheme has a natural scheme structure. To see this, cover X with open affine subsets $U_i = \operatorname{Spec} A_i \subset X$, so that $U = \bigcup_i U \cap U_i$. But $U \cap U_i \subset U_i$ is open in an affine scheme, therefore it is a union of principal open subsets $D(f_{ij}) \subset U_i$, for $f_{ij} \in A_i$. Thus U admits a covering $U = \bigcup_{i,j} D(f_{ij})$, and each $D(f_{ij})$ is an affine scheme by Proposition 3.1.58.

PROPOSITION 3.1.60. *Let $\phi : A \to B$ be a ring homomorphisms. Then the continuous map $f_\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ of Lemma 3.1.55 extends to a morphism of affine schemes $(f_\phi, f_\phi^\#)$ such that $f_\phi^\#(\operatorname{Spec} A) = \phi$.*

Proof. Set $f = f_\phi$. First of all, we have to construct the sheaf homomorphism

$$f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_* \mathscr{O}_{\operatorname{Spec} B}.$$

For $g \in A$, the preimage of the principal open $D(g) \subset \operatorname{Spec} A$ is

$$\begin{aligned} f^{-1}(D(g)) &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}(\mathfrak{q}) \in D(g)\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid g \notin \phi^{-1}(\mathfrak{q})\} \\ &= \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi(g) \notin \mathfrak{q}\} \\ &= D(\phi(g)). \end{aligned}$$

There is an induced commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\phi} & B & & \\ \downarrow & & \downarrow & & \\ A_g & \xrightarrow{\phi_g} & B_{\phi(g)} & & \frac{a}{g^r} \mapsto \frac{\phi(a)}{\phi(g)^r} \\ \| & & \| & & \\ \mathscr{O}_{\operatorname{Spec} A}(D(g)) & & \mathscr{O}_{\operatorname{Spec} B}(D(\phi(g))) & = & f_* \mathscr{O}_{\operatorname{Spec} B}(D(g)) \end{array}$$

allowing us to set $f^\#(D(g)) = \phi_g$. These morphisms are compatible with restrictions to smaller principal opens, thus they give rise to a morphism of $\mathcal{B}$-sheaves on $\operatorname{Spec} A$, which in turn uniquely determines a morphism of sheaves $f^\#$ by Proposition 2.7.9. If we take global sections of $f^\#$ (i.e. we evaluate it on $D(1) = \operatorname{Spec} A$), we get back our original map ϕ, by construction.

We are left with checking that $(f, f^\#)$ induces local homomorphisms of local rings at the level of stalks. Assume $y = f(x)$, where $x \in \operatorname{Spec} B$ corresponds to a prime ideal $\mathfrak{q} \subset B$ and $y \in \operatorname{Spec} A$ corresponds to $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \subset A$. Then the canonical map

$$A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}, \quad \frac{a}{s} \mapsto \frac{\phi(a)}{\phi(s)}$$

is a local homomorphism of local rings, which coincides with

$$f_x^\#: \mathscr{O}_{\operatorname{Spec} A, y} \to (f_* \mathscr{O}_{\operatorname{Spec} B})_y \to \mathscr{O}_{\operatorname{Spec} B, x}. \qquad \square$$

It is very easy to check that sending $A \mapsto \operatorname{Spec} A$ is a contravariant *functor* (cf. Definition A.1.6) from rings to affine schemes. Proposition 3.1.60 says 'what to do' on morphisms. The axioms defining a functor (identity goes to identity, and compositions are preserved) are elementary, and therefore left to the reader.

We can finally prove the main result of this chapter.

THEOREM 3.1.61. *The functor* Spec*, from rings to affine schemes, induces an equivalence*

$$\operatorname{Spec}: \mathsf{Rings}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff}, \quad A \mapsto \operatorname{Spec} A,$$

with inverse functor given by $X \mapsto \mathscr{O}_X(X)$. In particular, $\operatorname{Spec} \mathbb{Z}$ *is a final object in* Aff*.*

Proof. The Spec functor is essentially surjective, by definition of affine scheme. We need to show it is fully faithful (cf. Remark A.1.15). Set $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We claim that the inverse of the mapping

$$\operatorname{Hom}_{\mathsf{Rings}^{\mathrm{op}}}(B, A) \to \operatorname{Hom}_{\mathsf{Aff}}(X, Y), \quad \phi \mapsto f_\phi.$$

is the map

$$\rho_{X,Y}: \operatorname{Hom}_{\mathsf{Aff}}(X, Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, B) \tag{3.1.13}$$

sending $f: X \to Y$ to $f^\#(Y): A = \mathscr{O}_Y(Y) \to f_* \mathscr{O}_X(Y) = \mathscr{O}_X(X) = B$. We must show that (3.1.13) is bijective for any pair of affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Note that we already know that $\rho_{X,Y}$ is surjective thanks to Proposition 3.1.60.

Fix $f \in \operatorname{Hom}_{\mathsf{Aff}}(X, Y)$. Set $\phi = \rho_{X,Y}(f) = f^\#(Y)$. We know by Proposition 3.1.60 that ϕ gives rise to a morphism of affine schemes $f_\phi: X \to Y$ such that $\rho_{X,Y}(f_\phi) = \phi = \rho_{X,Y}(f)$. It is thus enough to show that $f = f_\phi$.

We need to show that f and f_ϕ are the same map set-theoretically, and, once we know this, that $f_x^\# = (f_\phi)_x^\#$ are the same as local homomorphisms of local rings, for every $x \in X$. This will imply that $f^\# = f_\phi^\#$ by Exercise 2.4.13. Let's start. Let $\mathfrak{q} \subset B$ be the prime ideal corresponding to $x \in X = \operatorname{Spec} B$, and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to $f(x) \in Y = \operatorname{Spec} A$. We have two commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ {\scriptstyle \mathrm{loc}_{\mathfrak{p}}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}_{\mathfrak{q}}} \\ A_{\mathfrak{p}} & \xrightarrow{\ f_x^{\#}\ } & B_{\mathfrak{q}} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ \phi\ } & B \\ {\scriptstyle \mathrm{loc}_{\phi^{-1}(\mathfrak{q})}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}_{\mathfrak{q}}} \\ A_{\phi^{-1}(\mathfrak{q})} & \xrightarrow{\ (f_\phi)_x^{\#}\ } & B_{\mathfrak{q}} \end{array}$$

that we need to show are the same. The very existence of $f_x^{\#}$ (fitting in the left diagram) implies that whenever $s \notin \mathfrak{p}$ one must have $\phi(s) \notin \mathfrak{q}$, i.e.

$$\phi(A \setminus \mathfrak{p}) \subset B \setminus \mathfrak{q},$$

which implies $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$. But the local condition $(f_x^{\#})^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = \mathfrak{p}A_{\mathfrak{p}}$ (combined with the classical correspondence between prime ideals in a ring and in a localisation of it, cf. Lemma B.4.6) implies $\phi^{-1}(\mathfrak{q}) = \phi^{-1}\mathrm{loc}_{\mathfrak{q}}^{-1}(\mathfrak{q}B_{\mathfrak{q}}) = \mathrm{loc}_{\mathfrak{p}}^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$.

Thus $f = f_\phi$ set-theoretically. However, there is only one possible commutative diagram as above: the one where the bottom map sends $a/s \mapsto \phi(a)/\phi(s)$. Thus $f_x^{\#} = (f_\phi)_x^{\#}$ as wanted. This concludes the proof that $\rho_{X,Y}$ is bijective.

The final statement now follows, since a ring A is a $\mathbb{Z}$-algebra $\mathbb{Z} \to A$ in a unique way (or, equivalently, $\mathbb{Z}$ is an initial object in Rings). □

Remark 3.1.62. The map (3.1.13) is functorial in the following sense: for any morphism of affine schemes $g\colon Z = \operatorname{Spec} C \to \operatorname{Spec} B = X$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Aff}}(X,Y) & \xrightarrow{\ \rho_{X,Y}\ } & \mathrm{Hom}_{\mathsf{Rings}}(A,B) \\ \downarrow{\scriptstyle f \mapsto f\circ g} & & \downarrow{\scriptstyle \phi \mapsto g^{\#}(X)\circ\phi} \\ \mathrm{Hom}_{\mathsf{Aff}}(Z,Y) & \xrightarrow{\ \rho_{Z,Y}\ } & \mathrm{Hom}_{\mathsf{Rings}}(A,C) \end{array}$$

commutes. This is just a rephrasing of the fact that morphisms of locally ringed spaces can be composed! In a little more detail, fix $f \in \mathrm{Hom}_{\mathsf{Aff}}(X,Y)$. The upper journey takes f to the map $g^{\#}(X) \circ f^{\#}(Y) \in \mathrm{Hom}_{\mathsf{Rings}}(A,C)$, whereas the lower journey takes f to $(f \circ g)^{\#}(Y)$. These maps are clearly the same, since $(f\circ g)^{\#}\colon \mathscr{O}_Y \to f_*g_*\mathscr{O}_Z$ is nothing but the composition $f_*g^{\#} \circ f^{\#}\colon \mathscr{O}_Y \to f_*\mathscr{O}_X \to f_*g_*\mathscr{O}_Z$.

3.1.8 Examples of affine schemes and their morphisms

In this section we collect some examples of affine schemes (and morphisms between them), besides those already considered in Section 3.1.3 at a purely topological level.

Recall that open (resp. closed) immersion of schemes are just open (resp. closed) immersions in the category of locally ringed spaces (cf. Definition 2.10.8). The next two examples are very important.

Key Example 3.1.63 (Principal open immersions)**.** Let A be a ring, $f \in A$. It follows from Proposition 3.1.58 and Definition 2.10.8 that the canonical morphism

$$\operatorname{Spec} A_f \to \operatorname{Spec} A$$

is an open immersion of affine schemes.

Key Example 3.1.64 (Closed immersions)**.** Let A be a ring, $I \subset A$ an ideal. Set $B = A/I$. The canonical surjection $\phi \colon A \twoheadrightarrow B$ determines, and is determined by, a morphism of affine schemes

$$i \colon \operatorname{Spec} B \to \operatorname{Spec} A$$

This morphism is a *closed immersion* according to Definition 2.10.8. Indeed, it is a homeomorphism onto $\mathrm{V}(I) \subset \operatorname{Spec} A$, and induces a surjective map of sheaves, because $i^{\#}(\mathrm{D}(g))$ is surjective for every $g \in A$ (it agrees with the canonical surjection $A_g \twoheadrightarrow B_{\phi(g)}$), and by Proposition 2.7.9 it is enough to check surjectivity on a base of open sets.

In what follows, we grant the following proposition, saying that *all* closed immersions into an affine scheme are as in Key Example 3.1.64.

PROPOSITION 3.1.65 ([11, Ch. 2, Prop. 3.20]). *Let $Y = \operatorname{Spec} A$ be an affine scheme, and let $\iota \colon Z \hookrightarrow Y$ be a closed immersion. Then Z is affine, and there is a unique ideal $I \subset A$ such that ι induces an isomorphism of schemes $Z \xrightarrow{\sim} \operatorname{Spec} A/I$.*

Example 3.1.66 (Many maps to the point!)**.** One is used to think that there is only one map $\bullet \to \bullet$. However, this is not necessarily true in algebraic geometry: think of the identity $\mathbb{C} \to \mathbb{C}$, which is different from complex conjugation $\mathbb{C} \to \mathbb{C}$. By Theorem 3.1.61, they give rise to different maps $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. In fact, there are *infinitely many* morphisms $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{C}$. The set $\operatorname{Hom}_{\mathsf{Rings}}(\mathbb{C},\mathbb{C})$ contains the Galois group of $\mathbb{C} \supset \mathbb{Q}$![4] Another example, in characteristic $p > 0$, is the *Frobenius morphism*, namely the map $\Phi_{\mathbb{F}} \colon \operatorname{Spec}\mathbb{F} \to \operatorname{Spec}\mathbb{F}$ induced by the field homomorphism $\phi_{\mathbb{F}} \colon \mathbb{F} \to \mathbb{F}$ sending $x \mapsto x^p$. A field $\mathbb{F}$ is *perfect* if and only if either $\mathbb{F}$ has characteristic 0 or $\phi_{\mathbb{F}}$ is surjective. In particular, the Frobenius morphism $\Phi_{\mathbb{F}}$ is an isomorphism if and only if $\mathbb{F}$ is perfect. For instance, all finite fields are perfect, but $\mathbb{F}_p(t)$ is not perfect, since t is (for degree reasons) not of the form $f(t)^p/g(t)^p$ for any two polynomials $f(t)$ and $g(t)$. In any case, $\phi_{\mathbb{F}} \neq \mathrm{id}_{\mathbb{F}}$, thus $\Phi_{\mathbb{F}} \neq \mathrm{id}_{\operatorname{Spec}\mathbb{F}}$.

Remark 3.1.67. Note that there is, on the other hand, *only one* morphism of schemes $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{R}$. This is because there is only one ring endomorphism $\mathbb{R} \to \mathbb{R}$, namely the identity. Can you prove it? (**Hint**: start by showing that a ring endomorphism $\mathbb{R} \to \mathbb{R}$ is increasing). Also, note that there is *no morphism* $\operatorname{Spec}\mathbb{R} \to \operatorname{Spec}\mathbb{C}$, since there is no ring homomorphism $\mathbb{C} \to \mathbb{R}$.

[4] The cardinality of the automorphism group of $\mathbb{C}$ is 2^c, where $c = 2^{\aleph_0}$. More generally, the cardinality of $\operatorname{Aut}(\mathbf{k})$ for $\mathbf{k}$ an *algebraically closed* field, is $2^{\operatorname{card}\mathbf{k}}$, see [2].

Example 3.1.68 (Dual numbers)**.** Let A be a ring. Thanks to Theorem 3.1.61, we finally have made rigorous our claim (formulated with $A = \mathbf{k}$)

$$\operatorname{Spec} A \neq \operatorname{Spec} A[t]/t^2$$

from Example 3.1.26! See Example 3.1.69 and Example 3.1.70 for generalisations.

Example 3.1.69 (Curvilinear schemes)**.** Let $n > 0$ be an integer, and set $A_n = \mathbf{k}[t]/t^n$. There is a closed immersion

$$\operatorname{Spec} A_n \hookrightarrow \mathbb{A}^1_{\mathbf{k}}.$$

Note that $\dim_{\mathbf{k}} A_n = n$. The affine scheme $\operatorname{Spec} A_n$ is called a *curvilinear scheme of length* n. All these schemes admit a (bijective) closed immersion $\operatorname{Spec} \mathbf{k} \hookrightarrow \operatorname{Spec} A_n$, which is never an isomorphism (unless $n = 1$). Thus $\operatorname{Spec} A_n$ can be seen as a 'thickening' of the origin in $\mathbb{A}^1_{\mathbf{k}}$.

Example 3.1.70 (Fat points)**.** Let $\mathbf{k}$ be an algebraically closed field, $(A, \mathfrak{m})$ a local artinian $\mathbf{k}$-algebra with residue field $A/\mathfrak{m} = \mathbf{k}$. Then $\operatorname{Spec} A$ is topologically just one (closed) point, corresponding to the maximal ideal $\mathfrak{m} \subset A$. For instance, consider $A = \mathbf{k}[x, y]/(x^2, xy, y^2)$. These affine schemes are called *fat points*. Each fat point has a *length*, namely the number $\dim_{\mathbf{k}} A = \dim_{\mathbf{k}} \mathscr{O}_X(X)$. For instance, $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$ has length 3, and is *not curvilinear*. The closed immersion $\operatorname{Spec} A/\mathfrak{m} \hookrightarrow \operatorname{Spec} A$ is a bijection (both schemes have precisely one point), but $\operatorname{Spec} A/\mathfrak{m} = \operatorname{Spec} \mathbf{k} \neq \operatorname{Spec} A$ as schemes. Fat points encode nontrivial information in their structure sheaves!

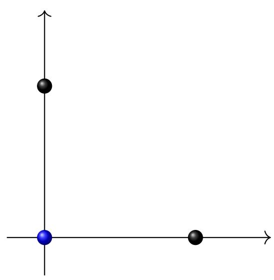


Figure 3.7: The length 3 fat point $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$ arises as the 'collision' of two points (black bullets) running towards the origin. It can be seen as a degeneration of the product ideal $(x-a, y)\cdot(x, y-b)$ for $a, b \to 0$.

Exercise 3.1.71. Show that the only fat point of length 3 which is not curvilinear is, up to isomorphism, precisely $\operatorname{Spec} \mathbf{k}[x, y]/(x^2, xy, y^2)$.

Example 3.1.72 (A non-affine scheme)**.** Let $X = \operatorname{Spec} \mathbb{Z}[x]$, and $z \in X$ the closed point corresponding to (p, x), where $p \in \mathbb{Z}$ is a prime number. Then $U = X \setminus \{z\} = \mathrm{D}(p) \cup \mathrm{D}(x)$

is not affine. Indeed, by Lemma 3.1.52(iii), we have $\mathscr{O}_X(U) \subset \mathscr{O}_X(\mathrm{D}(p)) \cap \mathscr{O}_X(\mathrm{D}(x)) = \mathbb{Z}[x, 1/p] \cap \mathbb{Z}[x, 1/x] = \mathbb{Z}[x] = \mathscr{O}_X(X)$, which readily implies $\mathscr{O}_X(U) = \mathscr{O}_X(X)$. Note that this example also shows that the union of two affine schemes need not be affine.

Example 3.1.73 (A non-affine scheme)**.** Let $n > 1$ be an integer, $\mathbf{k}$ a field, $A = \mathbf{k}[x_1, \ldots, x_n]$. Consider the origin of $X = \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} A$, namely the point $0 \in X$ corresponding to the maximal ideal $(x_1, \ldots, x_n) \subset A$. Form the open complement $U = X \setminus \{0\} \hookrightarrow X$. We now prove that the restriction map

$$\mathbf{k}[x_1, \ldots, x_n] = \mathscr{O}_X(X) \to \mathscr{O}_X(U)$$

is the identity. This proves that U is not affine, since U is not isomorphic to $\mathbb{A}^n_{\mathbf{k}}$. As before, we have $U = \bigcup_{1 \le i \le n} \mathrm{D}(x_i)$, so by Lemma 3.1.52(iii), we have

$$\mathscr{O}_X(U) \subset \bigcap_{1 \le i \le n} \mathrm{D}(x_i) = A_{x_1} \cap \cdots \cap A_{x_n}.$$

This can be proven directly to be equal to A. However, it also follows from the algebraic version of Hartog's Lemma below, combined with the fact that height 1 primes (see **??** for the definition of height of an ideal) in the (normal) domain A correspond to irreducible polynomials. The $n = 2$ case of this example can be seen as the geometric analogue of Example 3.1.72.

Lemma 3.1.74 ([11, Ch. 4, Lemma 1.13]). *Let A be a normal noetherian ring of dimension at least* 1*. Then*

$$A = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \operatorname{ht}(\mathfrak{p}) = 1}} A_{\mathfrak{p}},$$

the intersection being taken inside $\operatorname{Frac} A$*.*

Example 3.1.75 (Affine line minus one point)**.** If we take $n = 1$ in Example 3.1.73, we do in fact get an affine scheme $U = \mathbb{A}^1_{\mathbf{k}} \setminus \{0\} \hookrightarrow \mathbb{A}^1_{\mathbf{k}}$. Indeed, $U = \mathrm{D}(x) = \operatorname{Spec} \mathbf{k}[x]_x$. In fact, the ring $\mathbf{k}[x]_x = \{ f(x)/x^r \mid r \ge 0 \}$ is isomorphic to the $\mathbf{k}$-algebra $\mathbf{k}[x, x^{-1}] = \mathbf{k}[x, y]/(xy - 1)$, which yields a closed immersion $U \hookrightarrow \mathbb{A}^2_{\mathbf{k}}$.

Example 3.1.76 (Affine hypersurfaces)**.** Let $\mathbf{k}$ be a field, $f \in \mathbf{k}[x_1, \ldots, x_n]$. Then

$$Y_f = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n]/(f)$$

is called an *affine hypersurface* in $\mathbb{A}^n_{\mathbf{k}}$. The surjection $\mathbf{k}[x_1, \ldots, x_n] \twoheadrightarrow \mathbf{k}[x_1, \ldots, x_n]/(f)$ canonically determines a closed immersion

$$Y_f \hookrightarrow \mathbb{A}^n_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n].$$

Suppose (f) is a prime ideal in $\mathbf{k}[x_1, \ldots, x_n]$, so that $\mathbf{k}[x_1, \ldots, x_n]/(f)$ is an integral domain. Then (f) corresponds to the trivial (prime) ideal $(0) \subset \mathbf{k}[x_1, \ldots, x_n]/(f)$. This is the generic point of Y_f.

Example 3.1.77. As a special case of Example 3.1.76, consider $f = xy - z^2 \in \mathbb{C}[x,y,z]$. Its vanishing scheme

$$Y_f = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2) \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$$

is called the *affine quadric cone*.

Example 3.1.78. Let R be a DVR with fraction field K, and set $X = \operatorname{Spec} R = \{x_0, \xi\}$ where x_0 is the closed point. Then $K = \mathscr{O}_{X,\xi}$. The open immersion $\{\xi\} = X \setminus \{x_0\} \hookrightarrow X$ corresponds to the canonical inclusion $R \hookrightarrow K$.

Example 3.1.79. Let $\mu_n = \operatorname{Spec} \mathbf{k}[x]/(x^n - 1)$ for some $n > 1$. This is the scheme-theoretic version of the group of n-th roots of unity. One can prove that it is a group object in the category of $\mathbf{k}$-schemes. Such objects are called *algebraic groups*. As for μ_n, it comes with a natural closed immersion inside the affine line $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[x]$.

Example 3.1.80. Consider the morphism $f : \mathbb{A}^1_{\mathbf{k}} \to \mathbb{A}^1_{\mathbf{k}}$ defined by the ring homomorphism $\mathbf{k}[t] \to \mathbf{k}[t]$ sending $t \mapsto t^n$. This is the typical example of what we will call a *ramified* morphism. The intuition is the following: every point $x \in \mathbb{A}^1_{\mathbf{k}} \setminus 0$ in the target has precisely n preimages (because $\mathbf{k}$ is algebraically closed), but there is only one preimage over the origin $0 \in \mathbb{A}^1_{\mathbf{k}}$. Over this point, the morphism is 'fully ramified'. If we restrict f to $\mathbb{A}^1_{\mathbf{k}} \setminus 0 \to \mathbb{A}^1_{\mathbf{k}} \setminus 0$, it becomes *unramified*, and in fact *étale*. These notions are extremely important and will be treated in later chapters.

Example 3.1.81. The inclusion $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$ induces a morphism of affine schemes

$$\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}},$$

sending the generic point $(0) \subset \mathbb{C}[x]$ to the generic point $(0) \subset \mathbb{R}[x]$. For any $c \in \mathbb{R} \subset \mathbb{C}$, the maximal ideal $(x - c) \subset \mathbb{R}[x]$ is the preimage of the maximal ideal $(x - c) \subset \mathbb{C}[x]$, so $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ sends the closed point $(x - c) \in \mathbb{A}^1_{\mathbb{C}}$ to the closed point $(x - c) \in \mathbb{A}^1_{\mathbb{R}}$. On the other hand, if $c = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$, then both ideals

$$\mathfrak{p}_1 = (x - c),\ \mathfrak{p}_2 = (x - \overline{c}) \subset \mathbb{C}[x]$$

viewed as closed points of $\mathbb{A}^1_{\mathbb{C}}$, map to the closed point

$$\mathfrak{q} = (f) \in \mathbb{A}^1_{\mathbb{R}}, \quad f = (x - c)(x - \overline{c}).$$

However, note that this closed point has 'degree 2', for

$$\kappa(\mathfrak{q}) = \frac{\mathbb{R}[x]_{(f)}}{(f)\mathbb{R}[x]_{(f)}} \cong \frac{\mathbb{R}[x]}{(f)} \cong \mathbb{C},$$

since $\deg f = 2$. This does not happen for the other points $(x - c) \in \mathbb{A}^1_{\mathbb{R}}$, in the sense that

$$\kappa(x - c) = \frac{\mathbb{R}[x]_{(x-c)}}{(x - c)\mathbb{R}[x]_{(x-c)}} \cong \frac{\mathbb{R}[x]}{(x - c)} \cong \mathbb{R}.$$

As for $\xi = (0) \in \operatorname{Spec} \mathbb{R}[x]$, we have

$$\kappa(\xi) = \frac{\mathbb{R}[x]_{(0)}}{(0)} = \mathbb{R}[x]_{(0)} = \operatorname{Frac} \mathbb{R}[x] = \mathbb{R}(x).$$

We have used Exercise 3.1.30 in the three last displayed equations. The elements of $\kappa(\xi)$ are 'rational functions' g/h, not defined everywhere but *almost everywhere*, away from the (finitely many) zeros of $h \in \mathbb{R}[x]$.

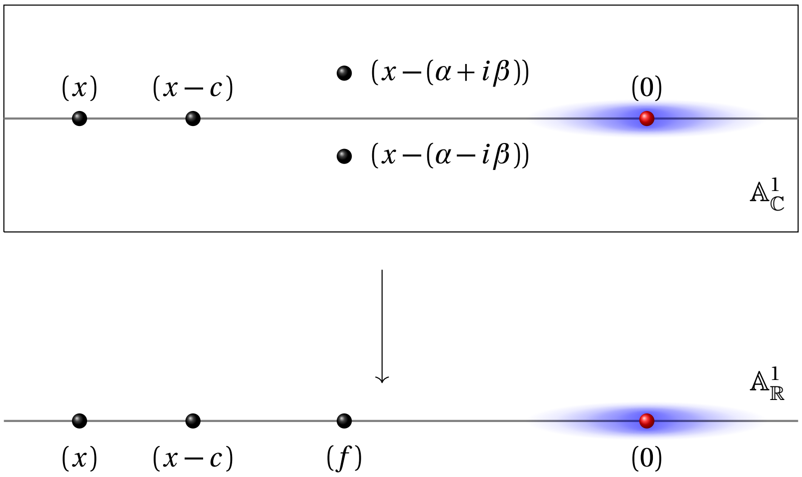


Figure 3.8: The morphism $\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{R}}$ induced by $\mathbb{R}[x] \hookrightarrow \mathbb{C}[x]$.

Example 3.1.82. This example is the arithmetic analogue of Example 3.1.81. Consider the inclusion of rings

$$\phi \colon \mathbb{Z} \hookrightarrow \mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1), \quad i^2 = -1.$$

Here $\mathbb{Z}[i]$ is the ring of *Gaussian integers*, which is an euclidean domain, in particular a principal ideal domain. We will recall some basic number theory in this example, in order to study the induced morphism

$$f \colon \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}.$$

The algebraic question is: what happens to a prime number $p \in \mathbb{Z}$ when one adds in the imaginary unit? More precisely, is the extension

$$(p) = p\mathbb{Z}[i] \subset \mathbb{Z}[i]$$

still a a prime ideal? If this happens we say that p is *inert*, otherwise that p *ramifies*. For sure $(p) \subset \mathbb{Z}[i]$ is still a principal ideal. By Fermat's theorem on sums of two squares, one has that $p > 2$ is a sum of squares if and only if $p \equiv 1 \bmod 4$. In this case, one can write $p = a^2 + b^2 = (a+ib)(a-ib)$ for some integers $a, b \in \mathbb{Z}$. Such primes then do ramify. On the other hand, if $p \equiv 3 \bmod 4$, then (p) *stays prime* in $\mathbb{Z}[i]$. Let us start with the smallest prime number: one has that $2=(1+i)(1-i)$, but $(1+i)=(1-i)$ as ideals in $\mathbb{Z}[i]$, since $i(1-i)=i+1$, thus $p=2$ ramifies. The next prime that ramifies is $5=(2+i)(2-i)$, followed by $13=(6+i)(6-i)$ (since 7 and 11 are inert). Primes (larger than 2) that ramify correspond, geometrically, to those points $(p)\in \operatorname{Spec}\mathbb{Z}$ having more than one preimage along f.

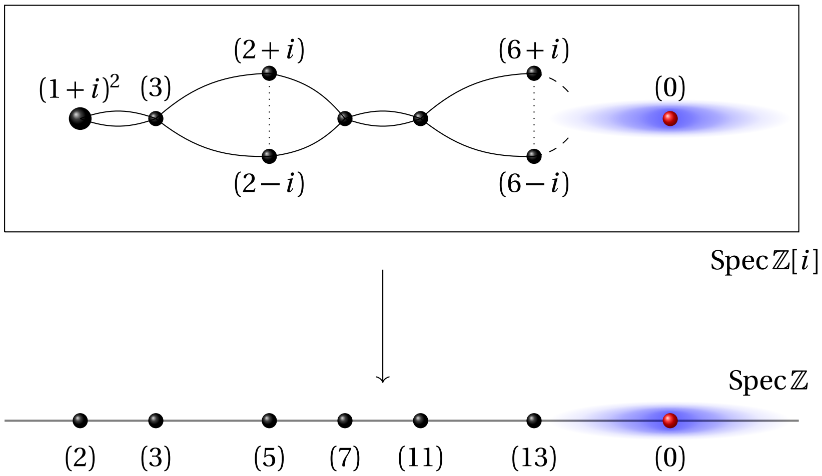


Figure 3.9: The morphism $f\colon \operatorname{Spec}\mathbb{Z}[i]\to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z}\hookrightarrow \mathbb{Z}[i]$.

Example 3.1.83 (The 'arithmetic surface')**.** Here is another arithmetic example. Consider the inclusion of rings $\mathbb{Z}\hookrightarrow \mathbb{Z}[x]$. We want to study the induced morphism

$$\operatorname{Spec}\mathbb{Z}[x]\to \operatorname{Spec}\mathbb{Z}.$$

A pictorial description of $\operatorname{Spec}\mathbb{Z}[x]$ was given by Mumford [13], see Figure 3.10.

First let us list all prime ideals in $\mathbb{Z}[x]$.

- $(0)\subset \mathbb{Z}[x]$ is a prime ideal, since $\mathbb{Z}[x]$ is an integral domain. It corresponds to the generic point of $\operatorname{Spec}\mathbb{Z}[x]$. The residue field if $\kappa(0)=\operatorname{Frac}\mathbb{Z}[x]=\mathbb{Q}(x)$.
- $(p)\subset \mathbb{Z}[x]$ is a prime ideal, for any prime number $p\in\mathbb{Z}$, since the quotient

  $$\mathbb{Z}[x]/(p)\cong \mathbb{F}_p[x]$$

  is an integral domain. These points are *not closed.* Note that each point (p) is precisely the generic point of the affine line

  $$\mathbb{A}^1_{\mathbb{F}_p}=\operatorname{Spec}\mathbb{F}_p[x]=\operatorname{Spec}\mathbb{Z}[x]/(p)\hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

  These lines are drawn as vertical lines in Figure 3.10, where they are denoted $\mathrm{V}(p)$. The residue field of $\operatorname{Spec}\mathbb{Z}[x]$ at these points is

  $$\kappa(p)=\mathbb{Z}[x]_{(p)}/(p)\mathbb{Z}[x]_{(p)}=\operatorname{Frac}\mathbb{Z}[x]/(p)=\operatorname{Frac}\mathbb{F}_p[x]=\mathbb{F}_p(x).$$

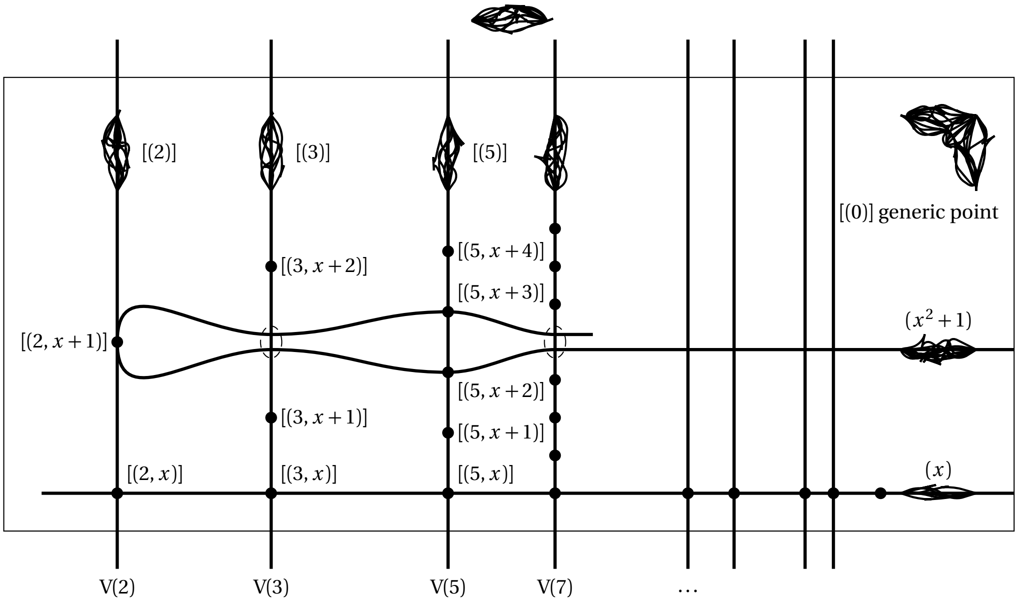


Figure 3.10: Picture's code is stolen from Pieter Belmans' website. This picture was originally drawn by David Mumford in [13], where he called $\operatorname{Spec}\mathbb{Z}[x]$ an *arithmetic surface*.

- $(f) \subset \mathbb{Z}[x]$, where $f \in \mathbb{Z}[x]$ is an irreducible polynomial (over $\mathbb{Z}$, hence over $\mathbb{Q}$ by Gauss' Lemma, hence one may even assume the gcd of its coefficients is equal to 1 after clearing denominators). Each such polynomial draws an "arithmetic curve"

$$\operatorname{Spec}\mathbb{Z}[x]/(f) \hookrightarrow \operatorname{Spec}\mathbb{Z}[x]$$

  depicted as a horizontal curve in Figure 3.10, where each such curve is denoted $\mathrm{V}(f)$. Clearly the point (f) is exactly the generic point of such arithmetic curve.

- $(p, f) \subset \mathbb{Z}[x]$, where p is a prime number and $f \in \mathbb{Z}[x]$ is an irreducible monic polynomial which stays irreducible over $\mathbb{F}_p$. These are all the closed points of $\operatorname{Spec}\mathbb{Z}[x]$. The residue fields of these points are finite extensions of $\mathbb{F}_p$.

Explicitly, one has

$$\mathrm{V}(p) = \left\{ (p), (p, f) \,\middle|\, f \text{ monic, irreducible over } \mathbb{Z} \text{ and } \mathbb{F}_p \right\}$$
$$\mathrm{V}(f) = \left\{ (f), (p, g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\},$$

and the intersection between a horizontal curve and a vertical line is

$$\mathrm{V}(f) \cap \mathrm{V}(p) = \left\{ (p, g) \,\middle|\, g \text{ divides } f \text{ modulo } p \right\}.$$

One such arithmetic curve $\mathrm{V}(f)$ is for instance the one "cut out by $x = 0$", consisting of a copy of $\operatorname{Spec}\mathbb{Z}$ itself, for

$$\mathrm{V}(x) = \operatorname{Spec}\mathbb{Z}[x]/(x) = \operatorname{Spec}\mathbb{Z} \hookrightarrow \operatorname{Spec}\mathbb{Z}[x].$$

Another curve is

$$V(x^2+1) = \operatorname{Spec} \mathbb{Z}[x]/(x^2+1) = \operatorname{Spec} \mathbb{Z}[i] \hookrightarrow \operatorname{Spec} \mathbb{Z}[x].$$

Let us now analyse Figure 3.10 carefully.

- V(2). Two "classical points" of this vertical line are the closed points $(2, x)$ and $(2, x+1)$, corresponding to the points with coordinates 0 and 1, respectively, in the affine line $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec} \mathbb{Z}[x]$. These two points are drawn as black bullets.

  But this affine line also intersects the arithmetic curve $V(x^2+1)$, since

  $$V(x^2+1) \cap V(2) = \{(2, x+1)\}.$$

  However, the point $(2, x+1)$ has 'multiplicity 2' since over $\mathbb{F}_2$ we have a splitting $x^2+1 = (x+1)(x+1)$. This is why the curve $V(x^2+1)$ is depicted tangent to the affine line V(2).

  Of course there are many other curves $V(f)$ meeting V(2). In other words, V(2) has many other points of the form $(2, f)$. They correspond to irreducible monic polynomials f which stay irreducible over $\mathbb{F}_2$. For instance,

  $$f = x^2 + x + 1$$

  has no roots over $\mathbb{F}_2$, and if we denote by α a root of f we have a splitting

  $$x^2 + x + 1 = (x+\alpha)(x+\alpha+1)$$

  over the larger field $\mathbb{F}_2[\alpha] = \{0, 1, \alpha, \alpha+1\} \supset \mathbb{F}_2$. We thus have two *different* residue fields

  $$\kappa(2, x+1) = \mathbb{F}_2$$
  $$\kappa(2, x^2+x+1) = \mathbb{F}_2[x]/(x^2+x+1) = \mathbb{F}_2[\alpha]$$

  for these two different types of points of $\mathbb{A}^1_{\mathbb{F}_2} \subset \operatorname{Spec} \mathbb{Z}[x]$.

- V(3). The polynomial x^2+1 is irreducible over $\mathbb{F}_3$ (having no roots), so the point $(3, x^2+1)$ is not a 'classical' point of $\mathbb{A}^1_{\mathbb{F}_3}$. Let α be a root of x^2+1. Then

  $$x^2 + 1 = (x-\alpha)(x-2\alpha)$$

  over $\mathbb{F}_3[\alpha] \supset \mathbb{F}_3$. In this larger field, the two points $(3, x-\alpha)$ and $(3, x-2\alpha)$ would be 'separated' and would be depicted as two classical points.

  The point $(3, x^2+1) = V(3) \cap V(x^2+1)$ is depicted as a small dotted circle. The curve $V(x^2+1)$ passes through this circle, but in the picture the two branches of the curve remain separated: this reflects the fact that the 'separation' of the roots happens over the larger field $\mathbb{F}_3[\alpha] \supset \mathbb{F}_3$. The residue field of $(3, x^2+1)$ is

$$\kappa(3, x^2+1) = \mathbb{F}_3[x]/(x^2+1) = \mathbb{F}_3[\alpha],$$

a degree 2 extension of $\mathbb{F}_3$.

- V(5). The polynomial x^2+1 factors as $(x+2)(x+3)$ over $\mathbb{F}_5$, so we have two classical points $(5, x+2)$ and $(5, x+3)$, both with residue field equal to $\mathbb{F}_5$.
- V(7). The situation here is similar to that of V(3).

Exercise 3.1.84. We have, in the previous example, unconsciously confirmed that $\mathbb{A}^1_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x]$ has way more that the 'traditional' p points $(x), (x-1), \ldots, (x-(p-1))$ corresponding to the coordinates $0, 1, \ldots, p-1 \in \mathbb{F}_p$. Show that this is always the case, by proving that $\mathbb{A}^1_{\mathbb{F}} = \operatorname{Spec} \mathbb{F}[x]$ is infinite for *any field* $\mathbb{F}$. (**Hint**: Euclid's proof of the infinitude of prime numbers!)

Example 3.1.85 (Nodal cubic)**.** Let $C = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^2(x+1)) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi : \mathbb{A}^1_{\mathbb{C}} \to C$$

induced by the ring homomorphism $\phi : \mathbb{C}[x, y]/(y^2 - x^2(x+1)) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2 - 1$ and $\phi(y) = t(t^2-1)$ is not an isomorphism. Indeed, the function $t = y/x$ is not regular at $(0,0)$, and as such it does not lie in the image of ϕ. There is no ring isomorphism $\mathbb{C}[t] \cong \mathbb{C}[x, y]/(y^2 - x^2(x+1))$. Note that f_ϕ is not even bijective on closed points: the origin $(0,0) \in C$ has two preimages, corresponding to $t = \pm 1$.

Example 3.1.86 (Cuspidal cubic)**.** Let $C = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3) \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$. Then the morphism

$$f_\phi : \mathbb{A}^1_{\mathbb{C}} \to C$$

induced by the ring homomorphism $\phi : \mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[t]$ defined by $\phi(x) = t^2$ and $\phi(y) = t^3$ is a bijective morphism, but not an isomorphism. It sends the closed point $(t-a) \in \mathbb{A}^1_{\mathbb{C}}$ to the point of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates (a^2, a^3). The morphism f_ϕ is called a *rational parametrisation* of the plane curve $C \hookrightarrow \mathbb{A}^2_{\mathbb{C}}$.

We have learnt from several examples (including Examples 3.1.66, 3.1.68, 3.1.85 and 3.1.86) that

a bijective morphism of schemes need not be an isomorphism.

Example 3.1.87. Consider the ring homomorphism

$$\phi : \mathbf{k}[x, y] \to \mathbf{k}[x, y, z]/(xz - y)$$

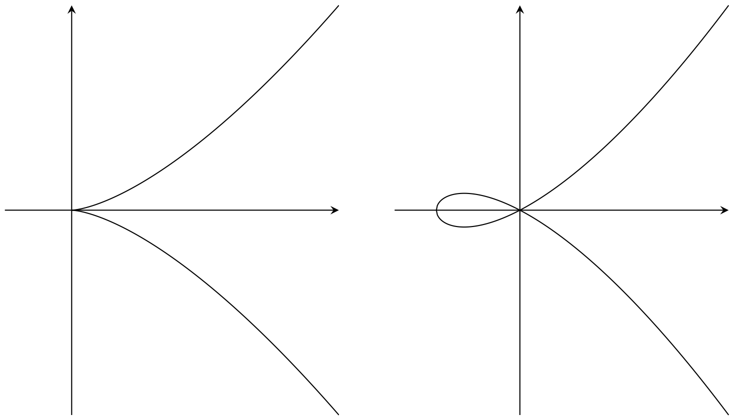


Figure 3.11: The (real points of the) cuspidal cubic curve $y^2 = x^3$ and the nodal cubic curve $y^2 = x^2(x+1)$.

sending $x \mapsto x$ and $y \mapsto y$. The corresponding morphism of affine schemes

$$f_\phi \colon \operatorname{Spec} \mathbf{k}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbf{k}}$$

sends $(a,ab,b) \mapsto (a,ab)$, and the generic point maps to generic point. The image of f_ϕ is $\mathrm{V}(x,y) \cup \mathrm{D}(x)$, which is neither open nor closed in $\mathbb{A}^2_{\mathbf{k}}$.

The image of the morphism f_ϕ in the previous example may look topologically weird. It is not that bad, though. Recall that a subset $T \subset X$ of a topological space X is *constructible* if it is a finite disjoint union of locally closed subsets. In general, a morphism of *algebraic varieties* (defined later in Important Definition 3.2.1 as those $\mathbf{k}$-schemes $X \to \operatorname{Spec} \mathbf{k}$ admitting an open cover by finitely many affine varieties) preserves constructible subsets.

THEOREM 3.1.88 (Chevalley). *Let $f: X \to Y$ be a morphism of algebraic varieties over $\mathbf{k}$. If $T \subset X$ is constructible, then $f(T) \subset Y$ is constructible. In particular, $f(X) \subset Y$ is constructible.*

3.2 Schemes

We already anticipated the definition of schemes in Important Definition 3.1.4, just because we could do so. Now we start with the general theory, but first we recall the definition verbatim.

Definition 3.2.1 (Scheme)**.** A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ in which every point $x \in X$ has an open neighbourhood $x \in U \subset X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

Keep also Terminology 3.1.33 and Definition 3.1.34 in mind.

3.2.1 The category of S-schemes and algebraic varieties

As we saw (cf. Notation 3.1.38), schemes form a category, denoted Sch. For any scheme S, we can form the *category* Sch_S *of S-schemes*, whose objects are pairs (X, f), where X is a scheme and $f: X \to S$ is a morphism of schemes. Morphisms

$$(X_1, f_1) \to (X_2, f_2)$$

in Sch_S are morphisms of schemes $g: X_1 \to X_2$ such that $f_2 \circ g = f_1$. We often call them *morphisms over S* or S-morphisms (or A-morphisms if $S = \operatorname{Spec} A$).

$$\begin{array}{ccc} X_1 & \xrightarrow{\quad g \quad} & X_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & S & \end{array}$$

If $S = \operatorname{Spec} A$ is affine, we simply write Sch_A instead of $\mathsf{Sch}_{\operatorname{Spec} A}$. For instance, when B is an A-algebra via a ring homomorphism $A \to B$, one says that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an A-scheme via the canonical scheme morphism attached to $A \to B$.

Important Definition 3.2.1 (Algebraic variety)**.** Let $\mathbb{F}$ be a field. An *algebraic variety* over a field $\mathbb{F}$ (or simply a $\mathbb{F}$-*variety*) is a $\mathbb{F}$-scheme which admits an open cover by finitely many affine varieties over $\mathbb{F}$.

Caution 3.2.2. Different authors give different definitions of algebraic variety. Other variants include: reduced scheme of finite type over a field, reduced separated[5] scheme of finite type over a field, integral scheme of finite type over a field. Note that, with our definitions, fat points (different from $\operatorname{Spec} \mathbf{k}$) are considered to be (affine) algebraic varieties, even though they are not reduced. On the other hand, the nodal cubic (**??**) and the cuspidal cubic (**??**) are algebraic varieties.

Caution 3.2.3. It is *not* true that if X is an algebraic variety, its ring of regular function $\mathscr{O}_X(X)$ is finitely generated! See [16].

Definition 3.2.4 (Closed subscheme)**.** A *closed subscheme* of a scheme X is an equivalence class of closed immersions $Z \hookrightarrow X$ of schemes into X. The equivalence relation says that $\iota: Z \hookrightarrow X$ is equivalent to $\iota': Z' \hookrightarrow X$ if there is an isomorphism $\alpha: Z \xrightarrow{\sim} Z'$ such that $\iota' \circ \alpha = \iota$ (in other words, α is an isomorphism in Sch_X).

Note the crucial difference between open subscheme (cf. Definition 3.1.36) and closed subscheme: a given open *subset* $U \subset X$ is given by default a well precise structure sheaf (making into a scheme, cf. Remark 3.1.59), namely $\mathscr{O}_X|_U$, whereas on a closed *subset* $Z \hookrightarrow X$ there are a pletora of possible scheme structures. Finally, note that

[5] This notion will be introduced in **??**.

we haven't defined a closed subscheme of X as a scheme Z together with a closed immersion: we have defined it to be an *equivalence class* of closed immersions, so that by Proposition 2.10.11 we have a precise correspondence between closed subschemes of a scheme X and ideal sheaves $\mathscr{I} \subset \mathscr{O}_X$. In the affine case, thanks to Proposition 3.1.65, we have the following: for any ring A, there is a bijection

$$\{\text{closed subschemes of } \operatorname{Spec} A\} \simeq \{\text{ideals } I \subset A\}. \tag{3.2.1}$$

Spoiler 3.2.5. We will see in **??** that for every scheme X there is a 'nicest' closed subscheme $X_{\mathsf{red}} \hookrightarrow X$, called the *reduction* of X, which is topologically the same as X and is the smallest with this property.

3.2.2 Morphisms to an affine scheme

A complete characterisation of morphisms *of affine schemes* was given, somewhat implicitly, in Theorem 3.1.61. Now we let X be an arbitrary scheme. Our goal is to characterise morphisms

$$X \to \operatorname{Spec} A.$$

We will show that the natural map

$$\rho_{X,Y}\colon \operatorname{Hom}_{\mathsf{Sch}}(X,Y) \to \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)), \tag{3.2.2}$$

already introduced in (3.1.13) in the affine case, is a bijection. The map works just as in the affine case: a morphism $f\colon X \to Y$ is sent to $f^\#(Y)\colon A = \mathscr{O}_Y(Y) \to f_*\mathscr{O}_X(Y) = \mathscr{O}_X(X)$. Functoriality also holds, i.e. the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{Sch}}(X,Y) & \xrightarrow{\rho_{X,Y}} & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \\ \Big\downarrow{\scriptstyle f \mapsto f\circ g} & & \Big\downarrow{\scriptstyle \phi \mapsto g^\#(X)\circ\phi} \\ \operatorname{Hom}_{\mathsf{Sch}}(Z,Y) & \xrightarrow{\rho_{Z,Y}} & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_Z(Z)) \end{array}$$

commutes for any morphism $g\colon Z \to X$ (for the same reason as in Remark 3.1.62).

We need the following preliminary result.

LEMMA 3.2.6. *Let X, Y be schemes. Then sending*

$$U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U,Y) \in \mathsf{Sets}$$

for each open subset $U \subset X$ defines a sheaf of sets on X.

Proof. By Remark 2.2.6, we need to verify that given an open subset $U \subset X$, an open cover $U = \bigcup_{i\in I} U_i$ and a collection of morphisms $f_i\colon U_i \to Y$ such that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ (as morphisms of schemes!) for every $(i,j) \in I \times I$, there exists a unique $f\colon U \to Y$ such that

$f|_{U_i} = f_i$. (We have used the usual notation $U_{ij} = U_i \cap U_j$). At the level of topological spaces, it is clear that there is a unique continous map $f: U \to Y$ with the required property. We need to extend it (uniquely) to a morphism of *schemes*. So we need a well defined sheaf homomorphism $f^{\#}: \mathscr{O}_Y \to f_* \mathscr{O}_U$. Let $V \subset Y$ be an open subset. We define $f^{\#}(V): \mathscr{O}_Y(V) \to \mathscr{O}_U(f^{-1}V)$ as follows.

First of all, each $f_i: U_i \to Y$ induces a map $\mathscr{O}_Y(V) \to \mathscr{O}_{U_i}(f_i^{-1}V) = \mathscr{O}_U(f_i^{-1}V)$. Moreover, $f^{-1}V = \bigcup_{i\in I} f_i^{-1}V$ is an open covering, and since $\mathscr{O}_U$ is a sheaf we have a diagram

$$\begin{array}{ccccc} & \swarrow\!\cdots & \mathscr{O}_Y(V) & & \\ & & \big\downarrow \tau & & \\ \mathscr{O}_U(f^{-1}V) & \longrightarrow & \prod_{i\in I} \mathscr{O}_U(f_i^{-1}V) & \overset{\mu}{\underset{\nu}{\rightrightarrows}} & \prod_{(i,j)\in I\times I} \mathscr{O}_U(f_i^{-1}V \cap f_j^{-1}V) \end{array}$$

where the bottom row is an equaliser sequence. Saying that $f_i|_{U_{ij}} = f_i|_{U_{ij}}$ is like saying that $\mu \circ \tau = \nu \circ \tau$, thus by the universal property of equalisers there is precisely one way to fill in the dotted arrow to $\mathscr{O}_U(f^{-1}V)$. This is the definition of $f^{\#}(V)$. □

Remark 3.2.7. The statement of Lemma 3.2.6 remains true for locally ringed spaces or, more generally, ringed spaces: we have not used the actual definition of schemes for its proof.

THEOREM 3.2.8. *Let X be a scheme, $Y = \operatorname{Spec} A$ an affine scheme. Then the canonical map* (3.2.2) *is bijective.*

Proof. Fix a covering $X = \bigcup_{i\in I} U_i$, where $\iota_i: U_i = \operatorname{Spec} B_i \hookrightarrow X$ is an affine open subset. Since $U \mapsto \operatorname{Hom}_{\mathsf{Sch}}(U, Y)$ is a sheaf on X (cf. Lemma 3.2.6), the natural map

$$\alpha: \operatorname{Hom}_{\mathsf{Sch}}(X, Y) \to \prod_{i\in I} \operatorname{Hom}_{\mathsf{Sch}}(U_i, Y)$$

is injective. We have a new diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{Sch}}(X,Y) & \xrightarrow{\rho_{X,Y}} & \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X)) \\ \alpha \big\downarrow & & \big\downarrow \phi \mapsto (\iota_i^{\#}(X)\circ\phi)_{i\in I} \\ \prod_{i\in I} \operatorname{Hom}_{\mathsf{Sch}}(U_i, Y) & \xrightarrow{\beta} & \prod_{i\in I} \operatorname{Hom}_{\mathsf{Rings}}(A, B_i) \end{array}$$

where β is a bijection as confirmed during the proof of Theorem 3.1.61. It follows that $\rho_{X,Y}$ is injective. We are left to prove its surjectivity. Fix $\phi \in \operatorname{Hom}_{\mathsf{Rings}}(A, \mathscr{O}_X(X))$, and consider its image $(\phi_i)_{i\in I} \in \prod_{i\in I} \operatorname{Hom}_{\mathsf{Rings}}(A, B_i)$. This corresponds to a unique tuple of morphisms $(f_i: U_i \to Y)_{i\in I}$. These have the property that $f_i|_V = f_j|_V$ for every affine open subset $V \subset U_i \cap U_j$. To see this, notice that for any $i \in I$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{Sch}}(U_i,Y) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\mathsf{Rings}}(A,B_i) \\ \big\downarrow & & \big\downarrow {\scriptstyle \psi\mapsto j_i^\#(U_i)\circ\psi} \\ \mathrm{Hom}_{\mathsf{Sch}}(V,Y) & \xrightarrow{\ \sim\ } & \mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(V)) \end{array}$$

where $j_i \colon V \hookrightarrow U_i$ is the open immersion. It is clear that the image of f_i in the set $\mathrm{Hom}_{\mathsf{Rings}}(A,\mathscr{O}_X(V))$ does not depend on i, being equal to the image of ϕ, namely its post-composition with $\mathscr{O}_X(X) \to \mathscr{O}_X(V)$. Therefore all the f_i map to the same element of $\mathrm{Hom}_{\mathsf{Sch}}(V,Y)$, which is what we wanted to confirm. It now follows from Lemma 3.2.6 that $(f_i \colon U_i \to Y)_{i\in I}$ glue to a (unique) morphism $f\colon X \to Y$, which by construction maps to ϕ via $\rho_{X,Y}$. Thus $\rho_{X,Y}$ is surjective. □

COROLLARY 3.2.9. *Let A be a ring. To give a scheme over* Spec A *is the same as to give a scheme $(X,\mathscr{O}_X)$ along with an A-algebra structure on $\mathscr{O}_X$.*

COROLLARY 3.2.10. *Let X be a scheme. There is a canonical morphism*

$$X \to \operatorname{Spec} \mathscr{O}_X(X),$$

called the affinisation morphism for X. And sometimes $\operatorname{Spec}\mathscr{O}_X(X)$ *is called the affinisation of X.*

Proof. Take $Y = \operatorname{Spec}\mathscr{O}_X(X)$ and consider the morphism corresponding to the identity $\mathrm{id} \in \mathrm{Hom}_{\mathsf{Rings}}(\mathscr{O}_X(X),\mathscr{O}_X(X))$ under $\rho_{X,Y}$. □

Remark 3.2.11. A possible translation of Theorem 3.2.8 is the following: if $\Gamma(-)$ denotes the functor taking a scheme X to the ring of its regular functions $\mathscr{O}_X(X)$, then the pair of functors $(\Gamma(-),\mathrm{Spec})$ is an adjoint pair on

$$\mathsf{Sch} \underset{\mathrm{Spec}}{\overset{\Gamma(-)}{\rightleftarrows}} \mathsf{Rings}^{\mathrm{op}}$$

where of course Spec is now viewed as the composition $\mathsf{Rings}^{\mathrm{op}} \xrightarrow{\sim} \mathsf{Aff} \hookrightarrow \mathsf{Sch}$.

Exercise 3.2.12. Confirm that $\operatorname{Spec}\mathbb{Z}$ is a final object in the category of schemes, so that (in the notation of Section 3.2.1) in particular $\mathsf{Sch} = \mathsf{Sch}_{\mathbb{Z}}$.

3.2.3 Glueing schemes

You may have encountered interesting spaces such as *projective spaces* or *Grassmannians* before. For example, projective n-space over a field $\mathbb{F}$ can be defined as follows: consider the scaling action of $\mathbb{F}^\times$ on $\mathbb{F}^{n+1}\setminus 0$, sending $v \mapsto \lambda v$ for $\lambda\in\mathbb{F}^\times$, and set

$$\mathbb{P}^n(\mathbb{F}) = (\mathbb{F}^{n+1}\setminus 0)/\mathbb{F}^\times.$$

This is all good in the topological (or smooth) category, however we cannot make such a definition in algebraic geometry. Quotients exist (sometimes) and their theory has now become classical, but they are delicate to deal with.

We shall see *two ways* to define projective space in algebraic geometry. The first one is by glueing schemes (along open immersions). We now describe this procedure in full generality.

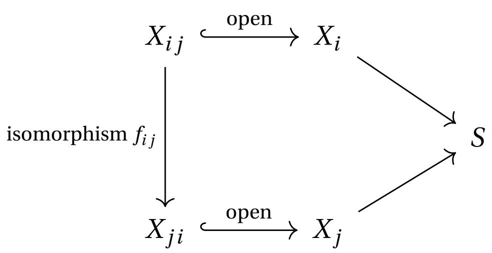
The input data are as follows:

(1) a scheme S,

(2) a family of S-schemes $\{X_i \to S \mid i \in I\}$,

(3) open subschemes $X_{ij} \subset X_i$ for every $(i,j) \in I \times I$,

(4) isomorphisms $f_{ij}\colon X_{ij} \xrightarrow{\sim} X_{ji}$ over S for every $(i,j) \in I \times I$.



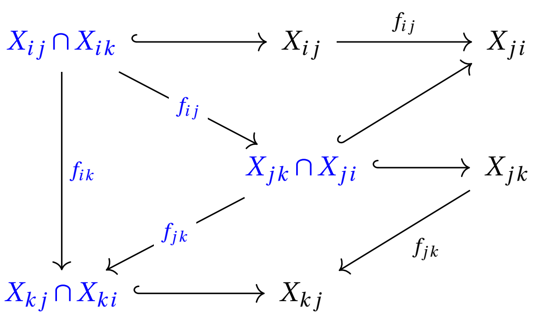
The assumptions on the input data are the following:

(i) $X_{ii} = X_i$ and $f_{ii} = \mathrm{id}_{X_i}$ for every $i \in I$,

(ii) $f_{ij}(X_{ij} \cap X_{ik}) = X_{jk} \cap X_{ji}$, for every $(i,j,k) \in I \times I \times I$

(iii) the *cocycle condition* holds: $f_{jk} \circ f_{ij} = f_{ik}$ on $X_{ij} \cap X_{ik}$.

The cocycle condition is the following compatibility:



THEOREM 3.2.13 (Glueing schemes). *Given the data (1)–(4) satisfying conditions (i)–(iii), there exists an S-scheme X (unique up to isomorphism), along with open immersions $\theta_i\colon X_i \hookrightarrow X$ over S such that $\theta_j|_{X_{ji}} \circ f_{ij} = \theta_i|_{X_{ij}}$ and $X = \bigcup_{i \in I} \theta_i(X_i)$. Moreover, $\theta_i(X_i) \cap \theta_j(X_j) = \theta_i(X_{ij})$.*

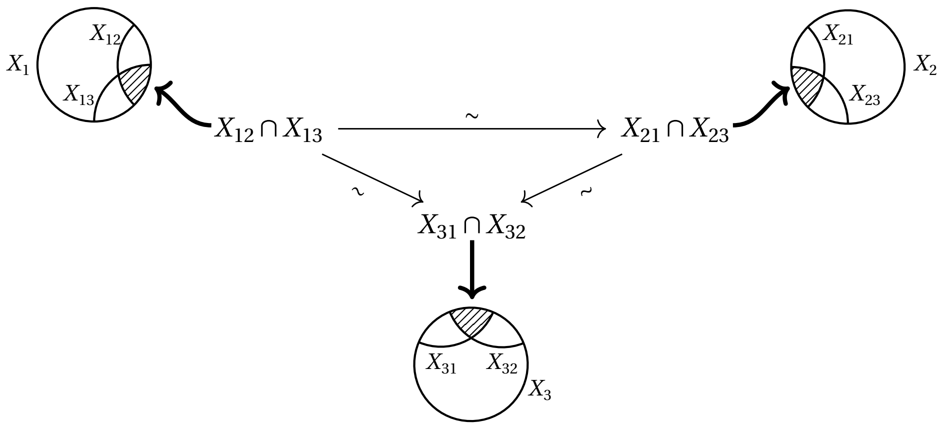


Figure 3.12: The glueing construction starting with 3 open subsets $X_1, X_2, X_3 \subset X$.

Proof. See [11, Ch. 2, Lemma 3.33]. □

Definition 3.2.14. The *disjoint union* of a family of S-schemes $\{X_i \to S\}_{i \in I}$ is the glueing of the family along $X_{ij} = \emptyset$ (and empty maps f_{ij}). It is denoted $\coprod_{i \in I} X_i$.

Next, we describe the construction of

$$\boxed{\mathbb{P}^n_A = \text{projective } n\text{-space over } A.}$$

Let A be a ring, $S = \operatorname{Spec} A$ the corresponding affine scheme, $n \geq 0$ an integer and $I = \{0, 1, \ldots, n\}$. Fix a variable x_i for every $i \in I$, and form the ring $R = A[x_0^{\pm}, x_1^{\pm}, \ldots, x_n^{\pm}]$. Consider the A-subalgebras

$$A_i = A\left[x_k x_i^{-1} \,\middle|\, 0 \leq k \leq n\right] \subset R, \quad i \in I.$$

Note that A_i is the homogeneous localisation of $A[x_0, x_1, \ldots, x_n]$ at the degree 1 element x_i, a polynomial ring in n variables (cf. Example 3.3.13). Each yields an A-scheme

$$X_i = \operatorname{Spec} A_i \to \operatorname{Spec} A, \quad i \in I.$$

Now, for each $j \neq i$, the scheme X_i contains the (principal) affine open subscheme

$$X_{ij} = \mathrm{D}(x_j x_i^{-1}) = \operatorname{Spec} (A_i)_{x_j x_i^{-1}} \subset X_i.$$

But

$$(A_i)_{x_j x_i^{-1}} = (A_j)_{x_i x_j^{-1}}$$

are *equal* as subrings of R, and thus we have canonical isomorphisms $f_{ij}\colon X_{ij} \xrightarrow{\sim} X_{ji}$. Explicitly, after the identifications

$$(A_i)_{x_j x_i^{-1}} \cong A_i[t]/(t \cdot x_j x_i^{-1} - 1)$$
$$(A_j)_{x_i x_j^{-1}} \cong A_j[u]/(u \cdot x_i x_j^{-1} - 1),$$

we see that an isomorphism between the quotient rings on the right hand sides is given by sending

$$x_j x_i^{-1} \mapsto u, \quad t \mapsto x_i x_j^{-1}, \quad x_k x_i^{-1} \mapsto x_k x_i^{-1} \text{ for } k \neq i, j.$$

The hypotheses of Theorem 3.2.13 are satisfied by our glueing data. The resulting A-scheme is called *projective n-space over A*, and is denoted $\mathbb{P}^n_A$. It has an open cover by $n+1$ affine open subsets isomorphic to affine spaces over A, namely $X_i = \operatorname{Spec} A_i \cong \mathbb{A}^n_A$. Indeed, the variables $\{ x_k x_i^{-1} \mid 0 \leq k \leq n \}$ are algebraically independent.

Remark 3.2.15. This construction shows that $\mathbb{P}^n_A$ is a quasicompact scheme. We shall see that it is not affine (unless $n = 0$).

Example 3.2.16 (Projective line)**.** The most explicit instance of the above construction of $\mathbb{P}^n_A$ arises for $n = 1$. In this case, our input data are simply two schemes $X_1 = \operatorname{Spec} A[t]$ and $X_2 = \operatorname{Spec} A[u]$, and the isomorphism $X_{12} = \mathrm{D}(t) \xrightarrow{\sim} \mathrm{D}(u) = X_{21}$ induced by the A-algebra isomorphism $A[u, u^{-1}] \xrightarrow{\sim} A[t, t^{-1}]$ sending $u \mapsto t^{-1}$. The glueing gives, by definition, the *projective line* $\mathbb{P}^1_A$. Similarly, Figure 3.12 can be seen as a pictorial construction of $\mathbb{P}^2_A$.

Example 3.2.17. Keep the notation of Example 3.2.16, but assume $A = \mathbf{k}$ is a field, for simplicity. Then, had we chosen the isomorphism $\mathbf{k}[u, u^{-1}] \xrightarrow{\sim} \mathbf{k}[t, t^{-1}]$ sending $u \mapsto t$, we would have of course identified the complements of the origin in the two affine lines, but we also would have 'kept the origin twice'. The result of the glueing is called an *affine line with double origin*. We shall come back to this scheme, for it is the prototypical example of a non-separated scheme. Separatedness is, as we shall see, the scheme-theoretic analogue of the Hausdorff property, which we have already given up on (cf. Remark 3.1.17). Since affine schemes are separated (cf. **??**), this also gives another example (besides Example 3.1.72 and Example 3.1.73) of a non-affine scheme.

Figure 3.13: The affine line with two origins.

Since the schemes X_i form an open covering of the glued up scheme X, by Example 2.3.4 we have an exact sequence of abelian groups

$$0 \to \mathscr{O}_X(X) \to \prod_{i \in I} \mathscr{O}_X(X_i) \to \prod_{(i,j) \in I \times I} \mathscr{O}_X(X_{ij}) \tag{3.2.3}$$

where the first map is restriction and the second map sends $(f_i)_i \mapsto (f_i|_{X_{ij}} - f_j|_{X_{ij}})_{i,j}$.

Exercise 3.2.18. Use the sequence (3.2.3) to show that $\mathscr{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) = A$. Observe, then, that $\mathbb{P}^n_A$ is not affine (unless $n = 0$)!

3.3 Projective schemes

In this section we define an important class of schemes, including *projective schemes*. These are the closed subschemes of $\mathbb{P}^n_A$ for some given ring A and some $n \geq 0$. They are the natural upgrade of classical *projective varieties* over a field. The general construction is somewhat analogous (though exhibiting many differences as well, see e.g. Remark 3.3.10, Caution 3.3.18 and Caution 3.3.28) to the construction of $\operatorname{Spec} A$ starting from a ring A. The main difference is that now we have to work with *graded rings*. The ubiquity of gradings and homogeneous ideals can be 'explained' at an informal level as follows: points $p = (a_0 : a_1 : \cdots : a_n)$ of classical projective space $\mathbb{P}^n(\mathbb{C})$ have *homogeneous coordinates*, meaning e.g. that $(1:2)$ is the same point as $(-3:-6)$ in $\mathbb{P}^1(\mathbb{C})$. As such, the evaluation of a polynomial $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$ at p is not well-defined. What *is* well-defined though, is the *vanishing* of f at p, so long as f is homogeneous, for

$$f(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_0, x_1, \ldots, x_n), \quad d = \deg f, \quad \lambda \in \mathbb{C}^\times.$$

We will see that the equation $f = 0$ defines a closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$. This will be called a *hypersurface in* $\mathbb{P}^n_{\mathbb{C}}$.

3.3.1 Zariski topology on Proj B

Let A be a ring. A *graded A-algebra* is an A-algebra $A \to B$ equipped with a decomposition

$$B = \bigoplus_{d \geq 0} B_d,$$

where $B_d \subset B$ are subgroups satisfying $B_d B_e \subset B_{d+e}$ for each $d, e \geq 0$, and such that the image of $A \to B$ is contained in B_0. In this situation, we have that

(1) $B_0 \subset B$ is a subring, so that B is naturally a B_0-algebra, and

(2) each graded piece B_d is a B_0-module.

Elements of B_d are called *homogeneous of degree d* (and $0 \in B$ is considered homogeneous of any degree). Every $f \in B$ has a unique decomposition $f = \sum_{0 \leq i \leq e} f_i$ into homogeneous elements f_i. An A-algebra homomorphism $\phi \colon B \to C$ is called a *graded homomorphism* if there exists an integer $e > 0$ such that $\phi(B_d) \subset C_{ed}$ for every $d \geq 0$. Graded A-algebras thus form a category. If B is a graded A-algebra, the ideal

$$B_+ = \bigoplus_{d > 0} B_d \subset B$$

is called the *irrelevant ideal*, for reasons that will become clear soon (cf. Remark 3.3.8).

Example 3.3.1. Let $B = A[x_0, x_1, \dots, x_n]$ be the polynomial ring with A-coefficients and with x_i in degree 1 for all i. Elements of B_d are simply the homogeneous polynomials of degree d in the classical sense. The irrelevant ideal is $B_+ = (x_0, x_1, \dots, x_n) \subset B$.

Let $I \subset B$ be an ideal. Then, the following are equivalent:

- $I \subset B$ is a graded submodule,
- I can be generated by homogeneous elements,
- $I = \bigoplus_{d \geq 0}(I \cap B_d)$,
- If $f \in I$ has homogeneous decomposition $f = f_0 + f_1 + \cdots + f_k$, then $f_e \in I$ for all e.

If any of these equivalent conditions is fulfilled, we say that I is *homogeneous*.

Remark 3.3.2. The class of homogeneous ideals in B is closed under sum, product, intersection, and radical.

Remark 3.3.3. Let $I \subset B$ be a homogeneous ideal. The quotient B/I is naturally a graded A-algebra via $(B/I)_d = B_d/(I \cap B_d)$.

Let $A \to B$ be a graded A-algebra. The localisation of B at a multiplicative subset $S \subset B$ inherits a grading as soon as S consists of homogeneous elements. If $\mathfrak{p} \subset B$ is a homogeneous prime ideal, we may localise B at

$$S(\mathfrak{p}) = \left\{ b \in B \setminus \mathfrak{p} \;\middle|\; b \text{ is homogeneous} \right\}.$$

This localisation, denoted $B_{\mathfrak{p}}$ with a slight abuse of notation (see also Warning B.4.9), contains as a subring its degree 0 piece, denoted $B_{(\mathfrak{p})}$. It is a local ring with maximal ideal

$$\mathfrak{m}_{(\mathfrak{p})} = \left\{ \frac{a}{h} \;\middle|\; a \in \mathfrak{p},\, h \in S(\mathfrak{p}),\, \deg a = \deg h \right\}.$$

The local ring $(B_{(\mathfrak{p})}, \mathfrak{m}_{(\mathfrak{p})})$ may be called the homogeneous localisation of B at $\mathfrak{p}$. Another key example of homogeneous localisation is the *homogeneous principal localisation*.

Construction 3.3.4 (Homogeneous principal localisation)**.** Let $A \to B$ be a graded A-algebra. If $f \in B$ is homogeneous of degree e, then $B_f = \bigoplus_{d \in \mathbb{Z}} (B_f)_d$, where

$$(B_f)_d = \left\{ \frac{a}{f^n} \;\middle|\; a \in B_{d+ne} \right\} \subset B_f.$$

Such graded rings are the only ones (that we consider) with negative graded pieces. We set

$$B_{(f)} = (B_f)_0 = \left\{ \frac{a}{f^n} \;\middle|\; a \in B_{ne} \right\}.$$

It is called the *homogeneous principal localisation* (or simply *homogeneous localisation*) of B at f. This is a ring by the condition (1), and in fact, it is an A-subalgebra of B_f, by our key assumption that $A \to B$ lands in B_0. This can be seen via the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B_0 & \xhookrightarrow{\text{subring}} & B \\ & & \Big\downarrow\vdots & & \Big\downarrow \ell \\ & & B_{(f)} & \hookrightarrow & B_f \end{array} \tag{3.3.1}$$

and the fact that ℓ preserves the grading. We endow $B_{(f)}$ with the trivial grading. The inclusion $B_{(f)} \hookrightarrow B_f$ turns B_f into a graded $B_{(f)}$-algebra, and this gives a natural morphism of affine schemes

$$\operatorname{Spec} B_f \longrightarrow \operatorname{Spec} B_{(f)}.$$

To an arbitrary ideal $I \subset B$ we may associate a homogeneous ideal $I^{\mathrm{h}} = \bigoplus_{d\geq 0}(I \cap B_d)$. Note that $I^{\mathrm{h}} \subset I$, with equality if and only if I is homogeneous.

LEMMA 3.3.5. *Let $I \subset B$ be a homogeneous ideal. Then I is prime if and only if whenever $ab \in I$ for homogeneous elements $a, b \in B$, one has that either $a \in I$ or $b \in I$.*

Proof. Let $a = \sum_{1\leq i\leq n} a_i$ and $b = \sum_{1\leq j\leq m} b_j$ be the homogeneous decompositions of two elements $a, b \in B$ such that $ab \in I$. Since I is homogeneous, it must contain all the homogeneous components of ab. Assume, by contradiction, that $a \notin I$ and $b \notin I$. Then, there is a largest d such that $a_d \notin I$ and a largest e such that $b_e \notin I$. We have $(ab)_{d+e} = \sum_{i+j=d+e} a_i b_j$, but every pair $(i, j) \neq (d, e)$ appearing in the sum satisfies either $i > d$ or $j > e$. Thus $a_i b_j \in I$ for every such pair. But since $(ab)_{d+e} \in I$ as well, we must have $a_d b_e \in I$. Thus, by our assumption, either $a_d \in I$ or $b_e \in I$. Contradiction. □

LEMMA 3.3.6. *If $I \subset B$ is a prime ideal, then the homogeneous ideal $I^{\mathrm{h}} \subset B$ is prime.*

Proof. We exploit Lemma 3.3.5. Let $a, b \in B$ be two homogeneous elements, say $a \in B_d$ and $b \in B_e$, such that $ab \in I^{\mathrm{h}}$. In fact, $ab \in I^{\mathrm{h}}_{d+e} = I \cap B_{d+e} \subset I$. Then, since I is prime, we have either $a \in I$ or $b \in I$. Thus either $a \in I \cap B_d \subset I^{\mathrm{h}}$, or $b \in I \cap B_e \subset I^{\mathrm{h}}$. □

Let $A \to B$ be a graded A-algebra. Define the *projective spectrum* of B to be the set

$$\boxed{\operatorname{Proj} B = \left\{ \mathfrak{p} \subset B \;\middle|\; \begin{array}{c} \mathfrak{p} \text{ is a homogeneous prime} \\ \text{ideal such that } \mathfrak{p} \not\supset B_+ \end{array} \right\}.}$$

Our goal is to put a structure of A-scheme on $\operatorname{Proj} B$. By Corollary 3.2.9, this amounts to construct a scheme $(\operatorname{Proj} B, \mathscr{O}_{\operatorname{Proj} B})$ along with an A-algebra structure on $\mathscr{O}_{\operatorname{Proj} B}$.

As in the affine case, we start from the topology on $\operatorname{Proj} B$. For a homogeneous ideal $I \subset B$, we define

$$\mathrm{V}_+(I) = \left\{ \mathfrak{p} \in \operatorname{Proj} B \;\middle|\; \mathfrak{p} \supset I \right\} \subset \operatorname{Proj} B.$$

These sets satisfy the axioms of closed subsets for a topology on $\operatorname{Proj} B$. The properties

(1) $V_+(I) \cup V_+(J) = V_+(I \cap J) = V_+(IJ)$

(2) $\bigcap_{\lambda \in \Lambda} V_+(I_\lambda) = V_+\left(\sum_{\lambda \in \Lambda} I_\lambda\right)$

(3) $V_+(B) = \emptyset$ and $V_+(0) = \operatorname{Proj} B$

are proved in a similar fashion to the affine case (cf. Lemma 3.1.6). The induced topology on $\operatorname{Proj} B$ is called the *Zariski topology*. Note that one also has

$$V_+(I) = V_+(\sqrt{I})$$

for any homogeneous ideal $I \subset B$.

LEMMA 3.3.7. *Let $I, J \subset B$ be homogeneous ideals.*

(i) $V_+(I) \subset V_+(J)$ *if and only if* $J \cap B_+ \subset \sqrt{I}$.

(ii) *One has* $\operatorname{Proj} B = \emptyset$ *if and only if* B_+ *is nilpotent.*

(iii) $V_+(I) = \emptyset$ *if and only if* $B_+ \subset \sqrt{I}$. *If* $\sqrt{I} = B_+$, *then* $V_+(J) = V_+(J \cap I)$.

Proof. We proceed step by step.

(i) Assume $J \cap B_+ \subset \sqrt{I}$, and fix a prime $\mathfrak{p} \in V_+(I)$. Then

$$\mathfrak{p} \supset \sqrt{I} \supset J \cap B_+ \supset JB_+,$$

and since $\mathfrak{p} \not\supset B_+$ we must have $\mathfrak{p} \in V_+(J)$.

Assume now that $V_+(I) \subset V_+(J)$. Recall that $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}$. Observe that any prime $\mathfrak{p} \in V(I)$ satisfies (since I is homogeneous) $I = I^{\mathrm{h}} \subset \mathfrak{p}^{\mathrm{h}}$. Now there are two possibilities. First: if $\mathfrak{p}^{\mathrm{h}} \not\supset B_+$, then by Lemma 3.3.6 $\mathfrak{p}^{\mathrm{h}}$ belongs to $V_+(I)$, thus by assumption $\mathfrak{p} \supset \mathfrak{p}^{\mathrm{h}} \supset J \supset J \cap B_+$, which implies $J \cap B_+ \subset \sqrt{I}$. Second option: if $\mathfrak{p}^{\mathrm{h}} \supset B_+$, we still have $\mathfrak{p} \supset \mathfrak{p}^{\mathrm{h}} \supset B_+ \supset J \cap B_+$. Thus, also in this case, we have $J \cap B_+ \subset \sqrt{I}$.

(ii) We have $\operatorname{Proj} B = \emptyset$ if and only if every homogeneous prime ideal $\mathfrak{p} \subset B$ contains B_+, i.e. $V_+(0) \subset V_+(B_+)$, i.e. $B_+ \subset \sqrt{0}$ by (i).

(iii) Follows from (i) applied to $V_+(I) \subset \emptyset = V_+(B_+)$. Finally, $V_+(J \cap I) = V_+(J) \cup V_+(I) = V_+(J) \cup V_+(B_+) = V_+(J)$. □

Remark 3.3.8. Condition (iii) explains why B_+ is called *irrelevant*: the operation $V_+(-)$ sends it to the empty set, and so does to all radical ideals that contain it. One should keep in mind the case $B = \mathbf{k}[x_0, x_1, \ldots, x_n]$, where $B_+ = (x_0, x_1, \ldots, x_n)$ should 'correspond to the origin'. But there is no origin in $\mathbb{P}^n(\mathbf{k})$!

Notation 3.3.9. Let $f \in B$ be a homogeneous element. We call $\mathrm{D}_+(f) = \operatorname{Proj} B \setminus \mathrm{V}_+(fB)$ a *principal open set* in $\operatorname{Proj} B$. We simply write $\mathrm{V}_+(f)$ instead of $\mathrm{V}_+(fB)$.

Note that, as in the affine case, we have the identity

$$\mathrm{D}_+(fg) = \mathrm{D}_+(f) \cap \mathrm{D}_+(g)$$

for any two homogeneous elements $f, g \in B_+$.

Remark 3.3.10. Note that $\operatorname{Proj} B$ need not be quasicompact. For instance,

$$\operatorname{Proj} \mathbb{Z}[x_1, x_2, \ldots] = \bigcup_{i \geq 1} \mathrm{D}_+(x_i)$$

is an open cover admitting no finite subcover. This may sound counterintuitive: projective things ‘should’ be compact, affine things should not. But the Zariski topology is funny. When we will have the correct notion of compactness, your intuition will get realigned.

There is a canonical inclusion

$$\varepsilon \colon \operatorname{Proj} B \hookrightarrow \operatorname{Spec} B$$

and for any $f \in B$ one has $\mathrm{V}(f) \cap \operatorname{Proj} B = \bigcap_{0 \leq i \leq e} \mathrm{V}_+(f_i)$ if $f = f_0 + f_1 + \cdots + f_e$ is the homogeneous decomposition of $f \in B$. Then $\mathrm{D}(f) \cap \operatorname{Proj} B = \bigcup_{0 \leq i \leq e} \mathrm{D}_+(f_i)$. This shows that the Zariski topology on $\operatorname{Proj} B$ is induced by the Zariski topology on $\operatorname{Spec} B$ (i.e. it agrees with the subspace topology), and moreover

$$\operatorname{Proj} B \setminus \mathrm{V}_+(I) = \bigcup_{\substack{f \in I \\ f \text{ homogeneous}}} \mathrm{D}_+(f)$$

for any homogeneous ideal $I \subset B$. In particular, the principal opens

$$\left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \,\middle|\, f \text{ is homogeneous} \right\}$$

form a base for the Zariski topology. In fact, one can focus only on those $\mathrm{D}_+(f)$ where $f \in B_+$. The reason is the following: suppose $B_+ = (f_i \mid i \in I)$ with f_i homogeneous. Then,

$$\operatorname{Proj} B = \operatorname{Proj} B \setminus \emptyset = \operatorname{Proj} B \setminus \mathrm{V}_+(B_+) = \operatorname{Proj} B \setminus \bigcap_{i \in I} \mathrm{V}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(f_i),$$

so that for any homogeneous $g \in B$ we have

$$\mathrm{D}_+(g) = \mathrm{D}_+(g) \cap \operatorname{Proj} B = \mathrm{D}_+(g) \cap \bigcup_{i \in I} \mathrm{D}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(g) \cap \mathrm{D}_+(f_i) = \bigcup_{i \in I} \mathrm{D}_+(g f_i),$$

where of course $g f_i \in B_+$ for every $i \in I$. We will thus use

$$\mathscr{B} = \left\{ \mathrm{D}_+(f) \subset \operatorname{Proj} B \,\middle|\, f \in B_+ \text{ is homogeneous} \right\} \tag{3.3.2}$$

as a base of open sets for $\operatorname{Proj} B$.

3.3.2 Structure sheaf on $\operatorname{Proj} B$

Let B be a graded A-algebra as in the previous section. We want to define a sheaf of A-algebras $\mathscr{O}_X$ on $X = \operatorname{Proj} B$, making $(X, \mathscr{O}_X)$ into an A-scheme. Our working definition will be

$$\mathrm{D}_+(f) \longmapsto B_{(f)}, \quad f \in B_+.$$

Here $B_{(f)}$ is the homogeneous principal localisation of Construction 3.3.4, which is an A-algebra by Diagram (3.3.1). In order to make sense of this and verify it is a $\mathcal{B}$-sheaf, we need some (algebraic) preparation.

LEMMA 3.3.11. *Let $f \in B_+$ be homogeneous of degree d. Set $B^{(d)} = \bigoplus_{e \geq 0} B_{de} \subset B$. Then, there is a ring isomorphism*

$$\alpha_f \colon B^{(d)}/(f-1)B^{(d)} \xrightarrow{\sim} B_{(f)}.$$

In particular, if $\deg f = 1$, *we have*

$$\alpha_f \colon B/(f-1)B \xrightarrow{\sim} B_{(f)}.$$

Proof. There is a surjective ring homomorphism $B^{(d)} \twoheadrightarrow B_{(f)}$ defined on homogeneous elements (and then extended additively) by sending $a \in B_{de}$ to a/f^e. This sends $f \in B_d = B_1^{(d)}$ to 1, so descends to a map α_f. The inverse is constructed as follows. Pick $w = z/f^n \in B_{(f)}$, so that z is homogeneous of degree dn. Send w to

$$\text{the image of } z \in B_{dn} \subset B^{(d)} \text{ along } B^{(d)} \twoheadrightarrow B^{(d)}/(f-1)B^{(d)}.$$

It is straightforward to check that this is well-defined, and is the inverse of α_f. □

Terminology 3.3.12. The ring $B^{(d)}$ is called the *d-th Veronese ring* attached to B. It is an A-subalgebra of B.

Example 3.3.13. Let $B = A[x_0, x_1, \ldots, x_n]$ and $f = x_i$, which has degree 1. Then, $B^{(1)} = B$ and Lemma 3.3.11 yields

$$A[x_0, \ldots, x_n]_{(x_i)} \cong A[x_0, \ldots, x_n]/(x_i - 1) \cong A[x_0, \ldots, \widehat{x}_i, \ldots, x_n].$$[6]

Let $\mathcal{B}$ be the base of the Zariski topology on $\operatorname{Proj} B$ as in Equation (3.3.2). Our next goal is to construct a $\mathcal{B}$-sheaf of rings on $X = \operatorname{Proj} B$. By Lemma 2.7.7, this will uniquely extend to a sheaf, which will be denoted $\mathscr{O}_X$.

[6] Notational warning: do not confuse $A[x_0, \ldots, x_n]_{(x_i)}$ (homogeneous localisation) with the localisation of $A[x_0, \ldots, x_n]$ at the prime ideal $(x_i) = x_i A[x_0, \ldots, x_n]$. Same potential problem when $(f) = fB \subset B$ is a prime ideal.

If $f \in B_+$ is homogeneous, we have $\mathrm{D}_+(f) = \mathrm{D}(f) \cap \operatorname{Proj} B$. We next prove a few crucial properties of the composition

$$\begin{aligned} \theta\colon \mathrm{D}_+(f) &\hookrightarrow \mathrm{D}(f) = \operatorname{Spec} B_f \longrightarrow \operatorname{Spec} B_{(f)} \\ \mathfrak{p} &\longmapsto \mathfrak{p}B_f \cap B_{(f)}. \end{aligned}$$

LEMMA 3.3.14 ((De)homogenisation). *Let $f \in B_+$ be a homogeneous element.*

(i) $\theta\colon \mathrm{D}_+(f) \to \operatorname{Spec} B_{(f)}$ *is a homeomorphism.*

(ii) *If* $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$ *and* $\alpha = g^{\deg f}/f^{\deg g} \in B_{(f)}$, *then* $\theta(\mathrm{D}_+(g)) = \mathrm{D}(\alpha)$.

(iii) *If g and α are as in (ii), then there is a canonical homomorphism $B_{(f)} \to B_{(g)}$ inducing a ring isomorphism*

$$(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}.$$

Proof. We proceed step by step.

(i)–(ii) The map θ is continuous, as we have already observed that the Zariski topology on $\operatorname{Proj} B$ is induced by that of $\operatorname{Spec} B$. We first need to prove it is bijective. Then, proving (ii) will show that it is open, hence a homeomorphism.

θ is injective: Suppose $\mathfrak{p}B_f \cap B_{(f)} = \mathfrak{p}'B_f \cap B_{(f)}$ for $\mathfrak{p}$, $\mathfrak{p}'$ two elements of $\mathrm{D}_+(f)$. Fix a homogeneous generator $b \in \mathfrak{p}$, so that $b^{\deg f}/f^{\deg b} \in \mathfrak{p}B_f \cap B_{(f)} \subset \mathfrak{p}'B_f$. Then $b^{\deg f} \in \mathfrak{p}'$, and since $\mathfrak{p}'$ is prime we deduce $b \in \mathfrak{p}'$. Hence $\mathfrak{p} \subset \mathfrak{p}'$. Exchanging the roles of $\mathfrak{p}$ and $\mathfrak{p}'$ we obtain equality.

θ is surjective: Fix $\mathfrak{q} \in \operatorname{Spec} B_{(f)}$. Define a homogeneous ideal $\mathfrak{p} \subset B$ by declaring that $x \in B_d$ lies in $\mathfrak{p}$ if and only if $x^{\deg f}/f^d \in \mathfrak{q} \subset B_{(f)}$. It is homogeneous by construction, and it is prime as well. Indeed, pick two homogeneous elements $x \in B_d$ and $y \in B_e$ such that $xy \in \mathfrak{p}_{d+e}$. This means that

$$\frac{(xy)^{\deg f}}{f^{d+e}} = \frac{x^{\deg f}}{f^d}\frac{y^{\deg f}}{f^e} \in \mathfrak{q}.$$

Then either $x^{\deg f}/f^d$ or $y^{\deg f}/f^e$ lies in $\mathfrak{q}$, because $\mathfrak{q}$ is prime. But by definition this means that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Moreover $f \notin \mathfrak{p}$, for otherwise we would have $1 \in \mathfrak{q}$. Finally, $\mathfrak{p}B_f \cap B_{(f)} = \mathfrak{q}$, which shows surjectivity.

(ii) holds: Fix $\mathfrak{p} \in \mathrm{D}_+(f)$. It is clear that $g \in \mathfrak{p}$ if and only if $\alpha \in \mathfrak{p}B_f \cap B_{(f)}$.

(iii) (sketch): If $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, by Lemma 3.3.7(i) we have $gB = gB \cap B_+ \subset \sqrt{fB}$, i.e. $g^r = fb$ for some $b \in B$ and some $r > 0$. We may assume b is homogeneous by replacing it with its component of degree $r \cdot \deg g - \deg f$. Then $B_{(f)} \to B_{(g)}$ is defined by sending $a/f^n \mapsto ab^n/g^{rn}$. We are then in the situation

$$\begin{array}{ccc} B_{(f)} & \xrightarrow{\ell} & (B_{(f)})_\alpha \\ \downarrow & \swarrow & \\ B_{(g)} & & \end{array}$$

where ℓ is the (classical) localisation at $\alpha \in B_{(f)}$. Since α is invertible in $B_{(g)}$, the dotted arrow can be completed to a solid one. We leave it as an exercise to show that this map is an isomorphism. □

We are ready for the main theorem of this section.

THEOREM 3.3.15. *Let B be a graded A-algebra. Then $X = \operatorname{Proj} B$ is canonically an A-scheme, with the property that the principal open subset $\mathrm{D}_+(f) \subset X$ is affine and isomorphic to $\operatorname{Spec} B_{(f)}$, for any homogeneous element $f \in B_+$. Moreover, the local ring $\mathscr{O}_{X,x}$ at a point $x \in X$ corresponding to a homogeneous prime $\mathfrak{p} \subset B$ is canonically isomorphic to the homogeneous localisation $B_{(\mathfrak{p})}$.*

Proof. For $f \in B_+$ a homogeneous element, define

$$\mathscr{O}_X(\mathrm{D}_+(f)) = B_{(f)}. \tag{3.3.3}$$

We first confirm that this prescription defines a $\mathcal{B}$-presheaf on X. This, in fact, follows at once by Lemma 3.3.14(iii), which shows that a canonical restriction map $\mathscr{O}_X(\mathrm{D}_+(f)) \to \mathscr{O}_X(\mathrm{D}_+(g))$ exists whenever $\mathrm{D}_+(g) \subset \mathrm{D}_+(f)$, and that $B_{(f)}$ and $B_{(g)}$ are canonically isomorphic as soon as $\mathrm{D}_+(g) = \mathrm{D}_+(f)$.

In fact, (3.3.3) defines a $\mathcal{B}$-*sheaf* on X. This can be confirmed via the equaliser sequence. What we need to show is that for any $f \in B_+$ and any open cover $\mathrm{D}_+(f) = \bigcup_{i \in I} \mathrm{D}_+(f_i)$, the sequence

$$\mathscr{O}_X(\mathrm{D}_+(f)) \longrightarrow \prod_{i \in I} \mathscr{O}_X(\mathrm{D}_+(f_i)) \rightrightarrows \prod_{(i,j) \in I \times I} \mathscr{O}_X(\mathrm{D}_+(f_i f_j))$$

is an equaliser diagram in the category of A-algebras. This can be rewritten as the sequence

$$B_{(f)} \longrightarrow \prod_{i \in I} B_{(f_i)} \rightrightarrows \prod_{(i,j) \in I \times I} B_{(f_i f_j)}. \tag{3.3.4}$$

Let us define

$$\alpha_i = f_i^{\deg f} / f^{\deg f_i}, \quad \alpha_{ij} = (f_i f_j)^{\deg f} / f^{\deg f_i f_j}$$

in $B_{(f)}$. Then, we know that

$$\theta(\mathrm{D}_+(f_i)) = \mathrm{D}(\alpha_i), \quad \theta(\mathrm{D}_+(f_i f_j)) = \mathrm{D}(\alpha_{ij})$$

by Lemma 3.3.14(ii). Since $\theta\colon \mathrm{D}_+(f)\to \operatorname{Spec} B_{(f)}$ is a homeomorphism, we have an open covering

$$\operatorname{Spec} B_{(f)} = \bigcup_{i\in I}\theta(\mathrm{D}_+(f_i)) = \bigcup_{i\in I}\mathrm{D}(\alpha_i).$$

In particular, we have an equaliser sequence

$$B_{(f)} \longrightarrow \prod_{i\in I}(B_{(f)})_{\alpha_i} \rightrightarrows \prod_{(i,j)\in I\times I}(B_{(f)})_{\alpha_{ij}}$$

in the category of A-algebras, because $\mathscr{O}_{\operatorname{Spec} B_{(f)}}$ is a sheaf of A-algebras. But thanks to Lemma 3.3.14(iii) this is precisely the sequence (3.3.4). Therefore $\mathscr{O}_X$ is a $\mathcal{B}$-sheaf.

Let $\mathscr{O}_X$ denote the induced sheaf of A-algebras. The stalks are local rings. Indeed, if $x\in X$ corresponds to a homogeneous prime ideal $\mathfrak{p}\subset B$, one has a canonical isomorphism

$$\mathscr{O}_{X,x} = \varinjlim_{\substack{f \text{ homogeneous}\\ f\notin\mathfrak{p}}} B_{(f)} \xrightarrow{\sim} B_{(\mathfrak{p})}.$$

The proof is identical to the one we gave for Spec (cf. Theorem 3.1.28(c)).

It follows that the pair $(X,\mathscr{O}_X)$ defines a locally ringed space. Now, the homeomorphism $\theta\colon \mathrm{D}_+(f)\to\operatorname{Spec} B_{(f)}$ extends to an isomorphism of locally ringed spaces

$$(\theta,\theta^\#)\colon (\mathrm{D}_+(f),\mathscr{O}_X|_{\mathrm{D}_+(f)}) \xrightarrow{\sim} (\operatorname{Spec} B_{(f)},\mathscr{O}_{\operatorname{Spec} B_{(f)}})$$

which shows that $(X,\mathscr{O}_X)$ is a scheme with the sought after property. In a little more detail, to construct $\theta^\#\colon \mathscr{O}_{\operatorname{Spec} B_{(f)}}\to\theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})$, we take a principal open $\mathrm{D}(\alpha)\subset\operatorname{Spec} B_{(f)}$ and since $\alpha\in B_{(f)}$ we may write it as $g^r/f^{\deg g}$, where $r=\deg f$. Therefore we can apply Lemma 3.3.14(iii), which gives the isomorphism

$$(B_{(f)})_\alpha \xrightarrow{\sim} B_{(g)}.$$

This is our $\theta^\#(\mathrm{D}(\alpha))$, which makes sense since

$$\theta_*(\mathscr{O}_X|_{\mathrm{D}_+(f)})(\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\theta^{-1}\mathrm{D}(\alpha)) = \mathscr{O}_X|_{\mathrm{D}_+(f)}(\mathrm{D}_+(g)) = \mathscr{O}_X(\mathrm{D}_+(g)) = B_{(g)}.$$

Finally, the A-scheme structure of X is given by the fact that each $B_{(f)}$ is naturally an A-algebra, combined with Corollary 3.2.9. □

Example 3.3.16. Let $B=A[x_0,x_1,\ldots,x_n]$, with the usual grading ($\deg x_i=1$ for all i). Then

$$\operatorname{Proj} A[x_0,x_1,\ldots,x_n] = \mathbb{P}^n_A$$

where projective n-space over A was defined via glueing in Section 3.2.3. The structure morphism $\mathbb{P}^n_A\to\operatorname{Spec} A$ allows one to think of $\mathbb{P}^n_A$ as a 'family of projective spaces' parametrised by the points of $\operatorname{Spec} A$.

Example 3.3.17. We have $\mathbb{P}^0_A=\operatorname{Proj} A[x]\xrightarrow{\sim}\operatorname{Spec} A$. Indeed, $\operatorname{Proj} A[x]=\mathrm{D}_+(x)=\operatorname{Spec} A[x]_{(x)}=\operatorname{Spec} A[x]/(x-1)\xrightarrow{\sim}\operatorname{Spec} A$, using Example 3.3.13 for the identification $A[x]_{(x)}=A[x]/(x-1)$.

3.3.3 Proj is not a functor

One may think that, in analogy with the case of the affine spectrum, sending $B \mapsto \operatorname{Proj} B$ could be a functor from graded A-algebras to schemes. This is not the case. In this section we discuss why this fails and to what extend it can be remedied.

Caution 3.3.18. Proj is *not a functor*! It is not true that a morphism of graded A-algebras $\phi\colon B \to C$ induces a morphism of A-schemes $\operatorname{Proj} C \to \operatorname{Proj} B$ sending $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$. The problem is that

$$\mathfrak{p} \not\supset C_+ \text{ does not imply } \phi^{-1}\mathfrak{p} \not\supset B_+$$

for a homogeneous prime ideal $\mathfrak{p} \subset C$. See, however, Proposition 3.3.20 for the closest to a functor one can get.

Example 3.3.19. If $\phi\colon B = \mathbf{k}[x_0, x_1] \hookrightarrow \mathbf{k}[x_0, x_1, x_2] = C$ is the natural inclusion, then $C_+ = (x_0, x_1, x_2) \not\subset \mathfrak{p} = (x_0, x_1) \in \operatorname{Proj} C$, but $\phi^{-1}\mathfrak{p} = (x_0, x_1) = B_+$. This is the only 'problematic' point.

PROPOSITION 3.3.20. *Let $\phi\colon B \to C$ be a graded morphism of graded A-algebras. Then there is a canonical morphism of schemes*

$$f\colon \operatorname{Proj} C \setminus \mathrm{V}_+(B_+C) \longrightarrow \operatorname{Proj} B, \quad \mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$$

such that for any homogeneous $h \in B_+$ we have $f^{-1}(\mathrm{D}_+(h)) = \mathrm{D}_+(\phi(h))$, and the induced morphism $\mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$ of affine schemes corresponds to the canonical restriction $B_{(h)} \to C_{(\phi(h))}$.

Proof. If $\mathfrak{p} \subset C$ is a homogeneous prime ideal, then $\phi^{-1}\mathfrak{p} \subset B$ is a homogeneous prime ideal. We have

$$B_+ \not\subset \phi^{-1}\mathfrak{p} \iff B_+C \not\subset \phi(\phi^{-1}\mathfrak{p}) \subset \mathfrak{p},$$

therefore the association $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ is well-defined precisely on the subset $\operatorname{Proj} C \setminus \mathrm{V}_+(B_+C) \subset \operatorname{Proj} C$, which, being open, has a canonical scheme structure inherited from $\operatorname{Proj} C$. Note that f is continuous, since the Zariski topology is induced by that of the affine spectrum, and f is the restriction of the same map $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$ going from $\operatorname{Spec} C$ to $\operatorname{Spec} B$.

We thus only need to construct the morphism at the level of structure sheaves. Since morphisms to a fixed target form a sheaf (cf. Lemma 3.2.6), it is enough to define the morphism on a base of open subsets of $\operatorname{Proj} C$. Consider principal open subsets

$$\mathrm{D}_+(h) \subset \operatorname{Proj} B, \quad \mathrm{D}_+(\phi(h)) \subset \operatorname{Proj} C, \quad h \in B_+.$$

As h runs in B_+, the opens $\mathrm{D}_+(h) \subset \operatorname{Proj} B$ cover the target $\operatorname{Proj} B$, and $\mathrm{D}_+(\phi(h)) \subset \operatorname{Proj} C$ cover $\operatorname{Proj} C \setminus \mathrm{V}_+(B_+C)$. In the commutative diagram

$$\begin{array}{ccc}
B & \xrightarrow{\ \phi\ } & C \\
{\scriptstyle \ell_h}\downarrow & & \downarrow{\scriptstyle \ell_{\phi(h)}} \\
B_h & \xrightarrow{\ \phi_h\ } & C_{\phi(h)} \\
\uparrow & & \uparrow \\
B_{(h)} & \dashrightarrow & C_{(\phi(h))}
\end{array}$$

the map ϕ_h is a graded morphism, therefore it preserves the degree 0 pieces, which induces $B_{(h)} \to C_{(\phi(h))}$. Taking Spec of this map recovers precisely the morphism of affine schemes $f_h\colon f^{-1}\mathrm{D}_+(h) = \mathrm{D}_+(\phi(h)) \to \mathrm{D}_+(h)$. These morphisms agree on the intersections (reason: the map $B_{hk} \to C_{\phi(hk)}$ agrees with both the localisation of $B_h \to C_{\phi(h)}$ and the localisation of $B_k \to C_{\phi(k)}$), and therefore glue to a global morphism f. □

Example 3.3.21 (Projection from a point)**.** In the situation of Example 3.3.19, the morphism we obtain applying Proposition 3.3.20 is

$$\mathbb{P}^2_{\mathbf{k}} \setminus \mathrm{V}_+(x_0, x_1) = \mathbb{P}^2_{\mathbf{k}} \setminus \{(0:0:1)\} \to \mathbb{P}^1_{\mathbf{k}}.$$

More generally, we have a morphism

$$\mathbb{P}^{n+1}_{\mathbf{k}} \setminus \{(0:\cdots:0:1)\} \to \mathbb{P}^n_{\mathbf{k}}$$

defined by the inclusion of graded $\mathbf{k}$-algebras $\mathbf{k}[x_0,\ldots,x_n] \hookrightarrow \mathbf{k}[x_0,\ldots,x_n,x_{n+1}]$. This is called *projection from a point*. It sends the closed point $(a_0:\cdots:a_n:a_{n+1})$ to the closed point $(a_0:\cdots:a_n)$.

Example 3.3.22. Consider the graded morphism $B = \mathbf{k}[x_0,x_1] \to \mathbf{k}[y_0,y_1] = C$ sending $x_i \mapsto y_i^n$ for $i = 0,1$. In this case, $\mathrm{V}_+(B_+C) = \mathrm{V}_+(y_0^n, y_1^n) = \mathrm{V}_+(y_0,y_1) = \emptyset$, so we get a well-defined morphism $\mathbb{P}^1_{\mathbf{k}} \to \mathbb{P}^1_{\mathbf{k}}$, which on closed points sends $(a_0:a_1) \mapsto (a_0^n : a_1^n)$.

Projective varieties

If $\phi\colon B \to C$ is a *surjective* graded morphism of graded A-algebras, we have $C_+ = \phi(B_+) = B_+C$, hence by Proposition 3.3.20 there is global A-morphism

$$f\colon \operatorname{Proj} C \to \operatorname{Proj} B,$$

locally given by the closed immersions

$$\operatorname{Spec} C_{(\phi(f))} \hookrightarrow \operatorname{Spec} B_{(f)}$$

induced by the natural surjections $B_{(f)} \twoheadrightarrow C_{(\phi(f))}$. Therefore f is a closed immersion of A-schemes. A special case of this will be recorded as the next corollary.

COROLLARY 3.3.23. *Let $I \subset B = A[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal, and let $\phi \colon B \to B/I$ be the canonical surjection. Then $\phi(B_+) = (B/I)_+$. Therefore, the* Proj *construction yields a closed immersion*

$$\operatorname{Proj} B/I \hookrightarrow \mathbb{P}^n_A = \operatorname{Proj} B$$

over $\operatorname{Spec} A$*, with image homeomorphic to* $\mathrm{V}_+(I) \subset \operatorname{Proj} B$*.*

Terminology 3.3.24. Algebras of the form $A[x_0, x_1, \ldots, x_n]/I$ as in Corollary 3.3.23 are called *homogeneous A-algebras*. A scheme that is isomorphic to a closed subscheme of $\mathbb{P}^n_A$, for some $n \geq 0$, is called a *projective scheme over A*.

The converse of Corollary 3.3.23 is the content of the following exercise.

Exercise 3.3.25. Let $X \hookrightarrow \mathbb{P}^n_A$ be a closed immersion. Show that there exists a homogeneous ideal $I \subset B = A[x_0, x_1, \ldots, x_n]$ such that X is isomorphic to $\operatorname{Proj} B/I$. Show, by exhibiting an example, that I is not unique with this property (and note the difference with the affine case, cf. Equation (3.2.1)).

Important Definition 3.3.1 (Projective variety)**.** A *projective variety* over a field $\mathbb{F}$ is a projective scheme over $\mathbb{F}$, i.e. a closed subscheme of $\mathbb{P}^n_{\mathbb{F}}$ for some n. An algebraic variety is called *quasiprojective* (resp. *quasiaffine*) if it admits an open immersion into a projective (resp. affine) variety.

A morphism $X \to Y = \operatorname{Spec} A$ is is said to be *projective* if it factors as a closed immersion $X \hookrightarrow \mathbb{P}^n_A$ followed by the canonical projection $\mathbb{P}^n_A \to Y$.

Remark 3.3.26. Sanity check: projective varieties are algebraic varieties in the sense of Important Definition 3.2.1, by the observation (cf. Remark 3.2.15) that $\mathbb{P}^n_{\mathbb{F}}$ is quasicompact.

Terminology 3.3.27. Let $B = \mathbf{k}[x_0, \ldots, x_n]$. Fix a homogeneous polynomial of degree d. The closed subscheme $X = \operatorname{Proj} B/(f) \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$ is called a hypersurface of degree d. If $d = 1$ (resp. $d = 2, 3, 4, 5, \ldots$), we say X is a *hyperplane* (resp. a *quadric*, a *cubic*, a *quartic*, a *quintic*... hypersurface).

Caution 3.3.28. It is not true that an isomorphism of schemes $\operatorname{Proj} B \cong \operatorname{Proj} C$ yields an isomorphism of graded A-algebras. For instance, one has $\operatorname{Proj} B \cong \operatorname{Proj} B^{(d)}$ for any $d \geq 1$, but B is not isomorphic to $B^{(d)}$ if $d > 1$.

Exercise 3.3.29. Show that there is no nonconstant morphism $\mathbb{P}^n_{\mathbf{k}} \to \mathbb{P}^m_{\mathbf{k}}$ if $m < n$.

3.3.4 Examples of projective schemes

Let $\mathbf{k}$ be a field. First of all, let us clarify the relationship between the scheme $\mathbb{P}^n_{\mathbf{k}}$ and classical projective space $\mathbb{P}^n(\mathbf{k}) = (\mathbf{k}^{n+1} \setminus 0)/\mathbf{k}^\times$. There is a set-theoretic map

$$\mathbb{P}^n(\mathbf{k}) \longrightarrow \mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \ldots, x_n] \tag{3.3.5}$$

sending a point $(a_0 : a_1 : \cdots : a_n)$ to the homogeneous prime ideal

$$(a_i x_j - a_j x_i \mid 0 \le i, j \le n) \subset \mathbf{k}[x_0, x_1, \ldots, x_n]. \tag{3.3.6}$$

Note that such ideal can be viewed as generated by the 2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0 & a_1 & \cdots & a_n \end{pmatrix}.$$

The map (3.3.5) is a bijection onto the set of closed points of $\mathbb{P}^n_{\mathbf{k}}$. The points of the form (3.3.6) will be referred to as the *classical points* of $\mathbb{P}^n_{\mathbf{k}}$.

Terminology 3.3.30. If we set $\mathbb{P}^n_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, \ldots, x_n]$, we call $(x_0, x_1, \ldots, x_n)$ the homogeneous coordinates of $\mathbb{P}^n_{\mathbf{k}}$. The closed point of $\mathbb{P}^n_{\mathbf{k}}$ corresponding to (3.3.6) is denoted $(a_0 : a_1 : \cdots : a_n)$. For instance, the 'classical point' $(0 : 0 : \cdots : 1) \in \mathbb{P}^n(\mathbf{k})$ corresponds to the ideal $(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{P}^n_{\mathbf{k}}$.

Remark 3.3.31. In the case of $\mathbb{P}^1_{\mathbf{k}}$, just as for $\mathbb{A}^1_{\mathbf{k}}$, there is only one nonclassical point, namely the point ξ corresponding to the trivial ideal $(0) \subset \mathbf{k}[x_0, x_1]$. It is the generic point of $\mathbb{P}^1_{\mathbf{k}}$, and one has $\kappa(\xi) = \mathbf{k}(t)$.

Remark 3.3.32. If $\mathfrak{p} \in X = \operatorname{Proj} B$, the stalk of the structure sheaf $\mathscr{O}_X$ at $\mathfrak{p}$ is the homogeneous localisation $B_{(\mathfrak{p})}$. But, if $U = \operatorname{Spec} R \subset X$ is any affine open neighbourhood of $\mathfrak{p}$, we clearly have

$$R_{\mathfrak{p}} = \mathscr{O}_{X,\mathfrak{p}} = B_{(\mathfrak{p})}.$$

However, R is a different ring, so we have to understand what ideal $\mathfrak{p} \subset B$ becomes when viewed in the ring R. We explain this via an example. Consider for instance the (closed) 'coordinate point'

$$z_i = (0 : \cdots : 0 : 1 : 0 : \cdots : 0) \in \mathbb{P}^n_{\mathbf{k}},$$

with 1 sitting in the $(i+1)$-st slot, corresponding to the homogeneous prime ideal

$$\mathfrak{p}_i = (x_0, \ldots, \widehat{x}_i, \ldots, x_n) \subset \mathbf{k}[x_0, x_1, \ldots, x_n].$$

Then, we have $z_i \in \mathrm{D}_+(x_i) = \operatorname{Spec} \mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)}$, and

$$\mathbf{k}[x_0, x_1, \ldots, x_n]_{(x_i)} \cong \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n]$$

by Lemma 3.3.11. Under this identification, z_i corresponds to the origin in

$$\operatorname{Spec} \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n],$$

which in turn corresponds to the ideal $\mathfrak{q}_i = (x_0, \ldots, \widehat{x}_i, \ldots, x_n)$. Therefore

$$\mathscr{O}_{\mathbb{P}^n_{\mathbf{k}}, z_i} = \mathbf{k}[x_0, x_1, \ldots, x_n]_{(\mathfrak{p}_i)} = \mathbf{k}[x_0, \ldots, \widehat{x}_i, \ldots, x_n]_{\mathfrak{q}_i},$$

which consists of fractions f/g of polynomials in n variables, where $g(0, \ldots, 0) \neq 0 \in \mathbf{k}$.

Terminology 3.3.33. Let V be a $\mathbf{k}$-vector space of dimension $n+1$. The symmetric algebra $\operatorname{Sym} V^\vee$ is the polynomial ring $\mathbf{k}[x_0, x_1, \ldots, x_n]$ (polynomial functions on V), and one defines

$$\mathbb{P}(V) = \operatorname{Proj} \operatorname{Sym} V^\vee.$$

This is the projective space attached to V, whose closed points correspond to lines in V. It is isomorphic to $\mathbb{P}^n_{\mathbf{k}}$.

Example 3.3.34 (Plane curves)**.** Consider the same polynomial $xy - z^2 \in \mathbb{C}[x, y, z]$ of Example 3.1.77. Note that it is homogeneous. Now, its vanishing locus $\mathrm{V}_+(xy - z^2)$ is the topological image of a closed immersion into $\mathbb{P}^2_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x, y, z]$, namely the morphism

$$\operatorname{Proj} \mathbb{C}[x, y, z]/(xy - z^2) \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$$

induced by the surjection $\mathbb{C}[x, y, z] \twoheadrightarrow \mathbb{C}[x, y, z]/(xy - z^2)$. In general, the vanishing scheme of a degree 2 homogeneous polynomial $f \in \mathbb{C}[x, y, z]$ is called a *plane conic*. The vanishing scheme of an arbitrary homogeneous polynomial of degree d is called a *plane curve of degree d*.

Exercise 3.3.35. Show that all nondegenerate plane conics $\operatorname{Proj} \mathbb{C}[x, y, z]/(f) \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$ are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ (**Hint**: Show that you can reduce to the normal form $f = xy - z^2$, or $f = x^2 + y^2 + z^2$ if you prefer; then show that this particular plane conic is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$). In fact, you may want to show this over an arbitrary algebraically closed field $\mathbf{k}$ of characteristic different from 2.

Example 3.3.36. Let B be a graded A-algebra. Fix $d > 0$. Consider the Veronese ring $B^{(d)} = \bigoplus_{e \geq 0} B^{(d)}_e$, defined by $B^{(d)}_e = B_{de}$. We have an inclusion

$$\phi \colon B^{(d)} \hookrightarrow B,$$

which is a graded homomorphism of A-algebras. We claim that ϕ induces an A-morphism

$$v_d \colon \operatorname{Proj} B \longrightarrow \operatorname{Proj} B^{(d)}.$$

We have

$$B^{(d)}_+ = \bigoplus_{e > 0} B_{de}.$$

To prove that v_d is well-defined, fix a point $\mathfrak{p} \in \operatorname{Proj} B$. Assume, by contradiction, that $\phi^{-1}\mathfrak{p} \supset B_+^{(d)}$. Then $\mathfrak{p} \supset \phi(\phi^{-1}\mathfrak{p}) \supset \phi(B_+^{(d)}) = B_+ \cap \phi(B^{(d)})$. If $a \in B_+$, then $a^d \in B_+ \cap \phi(B^{(d)}) \subset \mathfrak{p}$, which implies $a \in \mathfrak{p}$, and in turn $B_+ \subset \mathfrak{p}$, whence a contradiction. Thus v_d is globally well-defined by Proposition 3.3.20.

PROPOSITION 3.3.37. *Let B be a graded A-algebra, and fix $d > 0$. Then the morphism*

$$v_d \colon \operatorname{Proj} B \longrightarrow \operatorname{Proj} B^{(d)}$$

is an isomorphism of A-schemes.

Proof. We first confirm that v_d, sending $\mathfrak{p} \mapsto \mathfrak{p} \cap B^{(d)}$, is a homeomorphism, and then we show that it is an isomorphism on an open cover of the source.

To see that v_d is injective, first observe that if $I \subset B$ is homogeneous, then so is $I \cap B^{(d)}$, and moreover, if $\mathfrak{p} \in \operatorname{Proj} B$, then $\mathfrak{p} \supset I$ if and only if $\mathfrak{p} \cap B^{(d)} \supset I \cap B^{(d)}$. Therefore v_d is injective.

To see that v_d is surjective, fix $\mathfrak{q}' \in \operatorname{Proj} B^{(d)}$ and observe that $\mathfrak{q} = \mathfrak{q}' B \subset B$ is again homogeneous and satisfies $\mathfrak{q} \cap B^{(d)} = \mathfrak{q}'$ inside $B^{(d)}$. We now prove that the homogeneous ideal $\mathfrak{p} = \sqrt{\mathfrak{q}} \subset B$ is prime. By Lemma 3.3.5, it is enough to check the primality condition on homogeneous elements. So let us take $a \in B_m$ and $b \in B_n$ such that $ab \in \mathfrak{p}$. There is an integer $r > 0$ such that $(ab)^{rd} \in \mathfrak{q}'$, so either $a^{rd} \in \mathfrak{q}'$ or $b^{rd} \in \mathfrak{q}'$ (since $\mathfrak{q}' \subset B$ is prime). Therefore either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. So $\mathfrak{p}$ is prime. Since $\mathfrak{q}' \in \operatorname{Proj} B^{(d)}$ we have $\mathfrak{q}' \not\supset B_+^{(d)}$, which implies $\mathfrak{p} \not\supset B_+$. Since $\mathfrak{p} \cap B_+^{(d)} = \mathfrak{q}'$, we have that $\mathfrak{p} \in \operatorname{Proj} B$ is a preimage of $\mathfrak{q}'$, thus v_d is surjective.

To see that v_d is a homeomorphism, it is now enough to observe that

$$v_d(\mathrm{V}_+(I)) = \mathrm{V}_+(I \cap B_+^{(d)}),$$

for any homogeneous ideal $I \subset B$. finish

□

Example 3.3.38 (Veronese embedding)**.** Fix a pair of positive integers n, d. Set $B = \mathbf{k}[x_0, \ldots, x_n]$. We construct a closed immersion

$$v_{d,n} \colon \mathbb{P}^n_{\mathbf{k}} \hookrightarrow \mathbb{P}^N_{\mathbf{k}}, \quad N = \binom{n+d}{d} - 1$$

as follows. Let $\{m_0, \ldots, m_N\} \subset B_d$ be linearly independent monomials of degree d in the variables $x_0, \ldots, x_n$. In other words, fix the standard monomial basis of B_d. Fix indeterminates $w_0, \ldots, w_N$. Define a $\mathbf{k}$-algebra homomorphism

$$\mathbf{k}[w_0, \ldots, w_N] \longrightarrow B, \quad w_i \mapsto m_i.$$

This morphism has image the Veronese subalgebra $B^{(d)} \hookrightarrow B$, therefore the factorisation

$$\mathbf{k}[w_0, \ldots, w_N] \twoheadrightarrow B^{(d)} \hookrightarrow B$$

induces

$$v_{d,n}\colon \mathbb{P}^n_{\mathbf{k}} \xrightarrow{\sim} \operatorname{Proj} B^{(d)} \hookrightarrow \mathbb{P}^N_{\mathbf{k}}.$$

This is the d-th *Veronese embedding* of $\mathbb{P}^n_{\mathbf{k}}$. It is also called the *d-uple embedding* of $\mathbb{P}^n_{\mathbf{k}}$. The next two examples are important special cases.

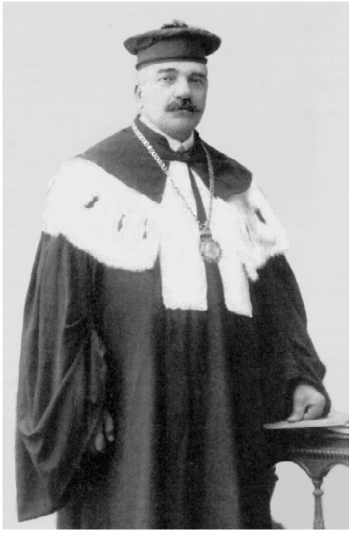


Figure 3.14: Giuseppe Veronese (1854–1917).

Example 3.3.39 (Rational normal curve)**.** Let $d > 0$ be an integer. The d-th Veronese embedding of $\mathbb{P}^1_{\mathbf{k}}$ is the map

$$\begin{aligned} \mathbb{P}^1_{\mathbf{k}} &\xhookrightarrow{v_{d,1}} \mathbb{P}^d_{\mathbf{k}} \\ (u:v) &\longmapsto (u^d : u^{d-1}v : \cdots : uv^{d-1} : v^d) \end{aligned}$$

defined by the map

$$\begin{aligned} \mathbf{k}[x_0, x_1, \ldots, x_d] &\longrightarrow \mathbf{k}[u,v] \\ x_i &\longmapsto u^{d-i}v^i. \end{aligned}$$

The image of this closed immersion is called the *rational normal curve* in $\mathbb{P}^d_{\mathbf{k}}$. If $d = 3$, the image of $\mathbb{P}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^3_{\mathbf{k}}$ is called a *twisted cubic in* $\mathbb{P}^3_{\mathbf{k}}$. Observe that the closed immersion defined above is cut out by the ideal J_d generated by the 2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}.$$

For instance, the twisted cubic is cut out by the ideal

$$J_3 = (x_0x_3 - x_1x_2,\, x_1^2 - x_0x_2,\, x_2^2 - x_1x_3) \subset \mathbf{k}[x_0, x_1, x_2, x_3].$$

Example 3.3.40. Set $d = n = 2$. The image of

$$v_{2,2}\colon \mathbb{P}^2_{\mathbf{k}} \hookrightarrow \mathbb{P}^5_{\mathbf{k}}$$

is called the *Veronese surface*. Consider the symmetric matrix

$$M = \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix}.$$

One may show that, as schemes,

$$v_{2,2}(\mathbb{P}^2_{\mathbf{k}}) = \operatorname{Proj} \mathbf{k}[w_0, \ldots, w_5]/J,$$

where J is the ideal generated by the 2-minors of M.

Example 3.3.41. Let B be a graded A-algebra with irrelevant ideal $B_+ \subset B$. Then, sending $\mathfrak{p} \mapsto \mathfrak{p}^{\mathrm{h}}$ defines a morphism of schemes

$$\operatorname{Spec} B \setminus \mathrm{V}(B_+) \to \operatorname{Proj} B.$$

For instance, we get a 'projection'

$$\mathbb{A}^{n+1}_A \setminus \{0\} \to \mathbb{P}^n_A.$$

If $\mathbf{k}$ is a field, this morphism is precisely, on closed points, the *quotient* by the scaling action of $\mathbf{k}^\times$ on $\mathbf{k}^{n+1} \setminus \{0\}$.

Example 3.3.42 (Projective closure)**.** If $A = \mathbf{k}[y_1, \ldots, y_n]$ and $B = \mathbf{k}[x_0, x_1, \ldots, x_n]$, we have (de)homogeneisation maps

$$\alpha\colon B^{\mathrm{h}} \to A, \quad g \mapsto g(1, y_1, \ldots, y_n)$$

and

$$\beta\colon A \to B^{\mathrm{h}}, \quad f \mapsto x_0^{\deg f} f(x_1/x_0, \ldots, x_n/x_0).$$

The open immersion

$$\iota_0\colon \mathbb{A}^n_{\mathbf{k}} = \mathrm{D}_+(x_0) \hookrightarrow \mathbb{P}^n_{\mathbf{k}}$$

allows one to turn an affine variety $Y = \operatorname{Spec} A/I \subset \mathbb{A}^n_{\mathbf{k}}$ into a projective variety $\overline{Y} \subset \mathbb{P}^n_{\mathbf{k}}$ by taking the closure along ι_0. It is a simple observation that the ideal $\overline{I} \subset B$ defining $\overline{Y}$ is nothing but the ideal generated by the image of I along the map β. However, it is *not* true that if $I = (f_1, \ldots, f_r)$ then $\overline{I} = (\beta(f_1), \ldots, \beta(f_r))$. For instance, consider the subvariety (isomorphic to $\mathbb{A}^1_{\mathbf{k}}$)

$$Y = \operatorname{Spec} \mathbf{k}[y_1, y_2, y_3]/(y_2 - y_1^2, y_3 - y_1^3) \subset \mathbb{A}^3_{\mathbf{k}}.$$

Its projective closure, i.e. its closure along the open immersion

$$\mathbb{A}^3_{\mathbf{k}} = \mathrm{D}_+(x_0) \hookrightarrow \mathbb{P}^3_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x_0, x_1, x_2, x_3],$$

agrees with the closure of the image of

$$\mathbb{A}^1_{\mathbf{k}} \hookrightarrow \mathbb{P}^1_{\mathbf{k}} \xhookrightarrow{\nu} \mathbb{P}^3_{\mathbf{k}},$$

where the first map sends $t \mapsto (1:t)$ and the map ν is the 3-uple embedding. In other words, $\overline{Y}$ is the twisted cubic in $\mathbb{P}^3_{\mathbf{k}}$. Thus $\overline{I} = (x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_0x_3)$, whereas $(\beta(y_2 - y_1^2), \beta(y_3 - y_1^3)) = (x_1^2 - x_0x_2, x_1^3 - x_0x_3)$.

Example 3.3.43. Recall the projection from a point introduced in Example 3.3.21. Consider the twisted cubic $Y \subset \mathbb{P}^3_{\mathbf{k}} = \operatorname{Proj} \mathbf{k}[x, y, z, w]$ as in Example 3.3.39. That is, Y is the curve in $\mathbb{P}^3_{\mathbf{k}}$ given parametrically by $(x, y, z, w) = (u^3, u^2v, uv^2, v^3)$. Let $P = (0:0:1:0) \in \mathbb{P}^3_{\mathbf{k}}$ and $H = \operatorname{Proj} \mathbf{k}[x, y, w] = \operatorname{Proj} \mathbf{k}[x, y, z, w]/(z) \subset \mathbb{P}^3_{\mathbf{k}}$ (identified topologically with $\mathrm{V}_+(z)$, and scheme-theoretically isomorphic to $\mathbb{P}^2_{\mathbf{k}}$). Then, projection from P is a morphism

$$\mathsf{pr}_P \colon \mathbb{P}^3_{\mathbf{k}} \setminus \{P\} \to H$$

sending $Y \subset \mathbb{P}^3_{\mathbf{k}} \setminus \{P\}$ to a *cuspidal plane cubic* $C \subset H$ (cf. Example 3.1.86). Indeed, $\mathsf{pr}_P \colon Y \to H$ sends $(u^3, u^2v, uv^2, v^3) \mapsto (u^3, u^2v, v^3)$. For instance, in the chart $v \neq 0$, we have

$$(u^2v)^3 = (u^3)^2v^3 \xrightarrow{v\neq 0} (u^2)^3 = (u^3)^2 \xrightarrow{\mathrm{X}=u^2,\mathrm{Y}=u^3} \mathrm{X}^3 = \mathrm{Y}^2.$$

But $(u^2v)^3 = (u^3)^2v^3$ holds globally, thus the image $\mathsf{pr}_P(Y)$ is exactly the plane curve $\mathrm{V}_+(y^3 - x^2w)$, the closure of the affine cuspidal cubic along the immersion $\mathrm{D}_+(w) \hookrightarrow \operatorname{Proj} \mathbf{k}[x, y, w] = \mathbb{P}^2_{\mathbf{k}}$.

A | Categories, functors, Yoneda Lemma

A.1 Minimal background on categories and functors

Definition A.1.1 (Category)**.** A *category* $\mathscr{C}$ is the datum of

(i) a class $\mathrm{Ob}(\mathscr{C})$ of 'objects',

(ii) a class $\mathrm{Hom}(\mathscr{C})$ of 'morphisms' (or 'arrows', or 'maps') between the objects,

(iii) class functions $d\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ and $t\colon \mathrm{Hom}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$ specifying domain and target of every morphism,

(iv) for each pair of objects x and y, a subclass $\mathrm{Hom}_{\mathscr{C}}(x,y) \subset \mathrm{Hom}(\mathscr{C})$ of morphisms with domain x and target y,

(v) a binary operation

$$\mathrm{Hom}_{\mathscr{C}}(x,y) \times \mathrm{Hom}_{\mathscr{C}}(y,z) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x,z),$$

called 'composition' of morphisms, for every triple of objects x, y and z.

Such data must fulfill the following axioms:

(CAT1) For every $x \in \mathrm{Ob}(\mathscr{C})$, there is an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{C}}(x,x)$ enjoying the properties

$$f \circ \mathrm{id}_x = f, \quad \mathrm{id}_y \circ g = g$$

for every morphism f with domain x, and for every morphism g with target y.

(CAT2) The associativity relation

$$(h \circ g) \circ f = h \circ (g \circ f)$$

holds for every triple (f,g,h) of composable morphisms.

Definition A.1.2 (Isomorphism)**.** Let $\mathscr{C}$ be a category, and fix two objects $x, y \in \mathrm{Ob}(\mathscr{C})$. An isomorphism between x and y is an invertible morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x,y)$, i.e. a morphism $f\colon x \to y$ such that there exists a morphism $g\colon y \to x$ satisfying $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. Two objects $x, y \in \mathrm{Ob}(\mathscr{C})$ are said to be *isomorphic* when there is an isomorphism $x \to y$ (often denoted '$x \xrightarrow{\sim} y$').

Definition A.1.3 (Small and locally small)**.** A category $\mathscr{C}$ is *small* if both $\mathrm{Ob}(\mathscr{C})$ and $\mathrm{Hom}(\mathscr{C})$ are sets, and not proper classes. We say that $\mathscr{C}$ is *locally small* if $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a set, and not a proper class, for every pair of objects x and y. For a locally small category $\mathscr{C}$, the sets $\mathrm{Hom}_{\mathscr{C}}(x, y)$ are called *hom-sets*.

Example A.1.4. The following are familiar examples of categories:

- Sets, the category of sets with morphisms the functions between sets,
- Grp, the category of groups with morphisms the group homomorphisms,
- Ab, the category of abelian groups with morphisms the group homomorphisms,
- Rings, the category of rings with morphisms the ring homomorphisms,
- Fields, the category of fields with morphisms the field homomorphisms,
- $\mathsf{Vec}_{\mathbb{F}}$, the category of vector spaces over a field $\mathbb{F}$ with morphisms the $\mathbb{F}$-linear maps,
- Alg_R, the category of algebras over a ring R, with morphisms the R-algebra homomorphisms,
- Top, the category of topological spaces, with morphisms the continuous maps,
- Mod_R, the category of modules over a ring R, with morphisms the R-linear maps,
- Mfd, the category of smooth manifolds, with morphisms the C^∞ maps.

Remark A.1.5. The category Sets is locally small, but not small (Russell's Paradox). The same is true, by the same argument, for all the categories in Example A.1.4.

Definition A.1.6 (Functor)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor from $\mathscr{C}$ to $\mathscr{C}'$, denoted $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$, is the assignment of

- an object $\mathsf{F}(x) \in \mathrm{Ob}(\mathscr{C}')$ for every $x \in \mathrm{Ob}(\mathscr{C})$, and
- a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$,

subject to the following axioms:

(1) $\mathsf{F}(\mathrm{id}_x) = \mathrm{id}_{\mathsf{F}(x)}$ for every $x \in \mathrm{Ob}(\mathscr{C})$,

(2) $\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$ for every pair (f, g) of composable arrows.

Remark A.1.7. By the axioms, functors preserve isomorphisms.

A functor as in Definition A.1.6 is said to be *covariant*. On the other hand, a *contravariant* functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ assigns a morphism $\mathsf{F}(f) \in \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(y), \mathsf{F}(x))$ for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(x, y)$, and condition (2) becomes $\mathsf{F}(g \circ f) = \mathsf{F}(f) \circ \mathsf{F}(g)$. For instance, taking a K-vector space V to its dual $V^* = \mathrm{Hom}_K(V, K)$ is a contravariant functor.

Example A.1.8. Every category $\mathscr{C}$ admits an *identity functor* $\mathrm{Id}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}$, sending every object and every morphism to itself.

Define $\mathscr{C}^{\mathrm{op}}$ to be the category with objects $\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C})$ and with

$$\mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathscr{C}}(y, x)$$

for every $x, y \in \mathrm{Ob}(\mathscr{C})$. Then a contravariant functor $\mathscr{C} \to \mathscr{C}'$ is the same as a covariant functor $\mathscr{C}^{\mathrm{op}} \to \mathscr{C}'$.

Definition A.1.9 (Natural transformation)**.** A *natural transformation* $\eta\colon \mathsf{F} \Rightarrow \mathsf{G}$ between two functors $\mathsf{F}, \mathsf{G}\colon \mathscr{C} \to \mathscr{C}'$ is the datum, for every $x \in \mathscr{C}$, of a morphism $\eta_x\colon \mathsf{F}(x) \to \mathsf{G}(x)$ in $\mathscr{C}'$, such that for every $f \in \mathrm{Hom}_{\mathscr{C}}(x_1, x_2)$ the diagram

$$\begin{array}{ccc} \mathsf{F}(x_1) & \xrightarrow{\eta_{x_1}} & \mathsf{G}(x_1) \\ {\scriptstyle \mathsf{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{G}(f)} \\ \mathsf{F}(x_2) & \xrightarrow{\eta_{x_2}} & \mathsf{G}(x_2) \end{array}$$

is commutative in $\mathscr{C}'$.

Definition A.1.10 (Natural isomorphism)**.** Let $\mathscr{C}, \mathscr{C}'$ be two categories. Let $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ be the category whose objects are functors $\mathscr{C} \to \mathscr{C}'$ and whose morphisms are the natural transformations. An isomorphism in the category $\mathsf{Fun}(\mathscr{C}, \mathscr{C}')$ is called a *natural isomorphism*.

Example A.1.11. Let K be a field, and $\mathscr{C}$ the category of finite dimensional K-vector spaces. Then we have two (covariant) functors $\mathscr{C} \to \mathscr{C}$, the former being the identity functor and the latter being the double dual functor, sending $V \mapsto V^{**}$. These two functors are naturally isomorphic.

Definition A.1.12 (Equivalence of categories)**.** Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. An *equivalence* between them is a pair of functors

$$\mathsf{F}\colon \mathscr{C} \to \mathscr{C}', \quad \mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$$

along with a pair of natural isomorphisms

$$\mathsf{F} \circ \mathsf{G} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}'}, \quad \mathsf{G} \circ \mathsf{F} \xrightarrow{\sim} \mathrm{Id}_{\mathscr{C}}.$$

Terminology A.1.13. One often says that a functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence when there exists a functor $\mathsf{G}\colon \mathscr{C}' \to \mathscr{C}$ along with a pair of natural isomorphisms as in Definition A.1.12.

Definition A.1.14 (Fully faithful, essentially surjective)**.** A (covariant) functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ is called:

(i) *fully faithful* if for any two objects $x, y \in \mathscr{C}$ the map of sets

$$\mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}'}(\mathsf{F}(x), \mathsf{F}(y))$$

is a bijection.

(ii) *essentially surjective* if every object of $\mathscr{C}'$ is isomorphic to an object of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$.

The following observation is quite useful.

Remark A.1.15. A fully faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathscr{C}'$ induces an equivalence of $\mathscr{C}$ with the essential image of F, namely the full subcategory of $\mathscr{C}'$ consisting of objects isomorphic to objects of the form $\mathsf{F}(x)$ for some $x \in \mathscr{C}$. Put differently, a functor induces an equivalence if and only if it is fully faithful and essentially surjective.

Definition A.1.16 (Concrete category)**.** A *concrete category* is a category $\mathscr{C}$ that is equipped with a faithful functor $\mathsf{F}\colon \mathscr{C} \to \mathsf{Sets}$ to the category of sets.

Note that concreteness is not a property, but rather an additional structure present on the category.

Another notion that is rather important in category theory is that of an adjoint pair of functors.

Definition A.1.17 (Adjoint pair)**.** Let $\mathscr{C}$ and $\mathscr{D}$ be (locally small) categories. Let $\mathsf{F}\colon \mathscr{C} \to \mathscr{D}$ and $\mathsf{G}\colon \mathscr{D} \to \mathscr{C}$ be functors. We say that (F, G) is an *adjoint pair* of functors if for every pair of objects $(c, d) \in \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$ one has a bijection of sets

$$\mathrm{Hom}_{\mathscr{D}}(\mathsf{F}(c), d) \cong \mathrm{Hom}_{\mathscr{C}}(c, \mathsf{G}(d)),$$

natural in both c and d. We say, more precisely, that F is a *left adjoint* to G and that G is a *right adjoint* to F.

Sometimes, one uses the pictorial description

$$\mathscr{C} \underset{\mathsf{G}}{\overset{\mathsf{F}}{\rightleftarrows}} \mathscr{D}$$

to say that (F, G) is an adjoint pair.

Example A.1.18. Here are some examples of adjunctions.

(a) Let $\mathsf{F}\colon \mathsf{Sets} \to \mathsf{Grp}$ be the functor sending a set S to the free group generated by the element of S. Let $\Phi\colon \mathsf{Grp} \to \mathsf{Sets}$ be the forgetful functor. Then (F, Φ) is an adjoint pair.
$$\mathsf{Sets} \underset{\Phi}{\overset{\mathsf{F}}{\rightleftarrows}} \mathsf{Grp}$$

(b) Let $\mathsf{j}\colon \mathsf{Ab} \hookrightarrow \mathsf{Grp}$ be the inclusion. It is right adjoint to the abelianisation functor $\mathsf{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ sending a group G to $G^{\mathrm{ab}} = G/[G,G]$. That is, $(\mathsf{ab}, \mathsf{j})$ is an adjoint pair.
$$\mathsf{Grp} \underset{\mathsf{j}}{\overset{\mathsf{ab}}{\rightleftarrows}} \mathsf{Ab}$$

(c) Let R be a ring. Consider the functor $\mathsf{sym}_R\colon \mathsf{Mod}_R \to \mathsf{Alg}_R$ sending $M \mapsto \mathsf{Sym}_R(M)$. Consider the forgetful functor $\Phi_R\colon \mathsf{Alg}_R \to \mathsf{Mod}_R$ sending an R-algebra to its underlying R-module. Then (sym_R, Φ_R) is an adjoint pair.
$$\mathsf{Mod}_R \underset{\Phi_R}{\overset{\mathsf{sym}_R}{\rightleftarrows}} \mathsf{Alg}_R$$

(d) Let $\alpha\colon R \to S$ be a ring homomorphism. Then every S-module is naturally an R-module, thus we have a forgetful functor $\Phi_\alpha\colon \mathsf{Mod}_S \to \mathsf{Mod}_R$. On the other hand, we have a functor (called extension of scalars) $-\otimes_R S\colon \mathsf{Mod}_R \to \mathsf{Mod}_S$ sending an R-module M to the S-module $M \otimes_R S$. Then $(-\otimes_R S, \Phi_\alpha)$ is an adjoint pair.
$$\mathsf{Mod}_R \underset{\Phi_\alpha}{\overset{-\otimes_R S}{\rightleftarrows}} \mathsf{Mod}_S$$

(e) Let ID be the category of integral domains (with morphisms the injective ring homomorphisms), Fields the category of fields. We have a functor $\mathsf{frac}\colon \mathsf{ID} \to \mathsf{Fields}$ sending a domain to its fraction field, and an inclusion functor $\mathsf{j}\colon \mathsf{Fields} \hookrightarrow \mathsf{ID}$. Then $(\mathsf{frac}, \mathsf{j})$ is an adjoint pair.
$$\mathsf{ID} \underset{\mathsf{j}}{\overset{\mathsf{frac}}{\rightleftarrows}} \mathsf{Fields}$$

(f) Let R be a ring, $M \in \mathsf{Mod}_R$ an R-module. Consider the endofunctors on the category Mod_R given by $-\otimes_R M$ and $\mathsf{h}^M = \mathrm{Hom}_{\mathsf{Mod}_R}(M,-)$. Then $(-\otimes_R M, \mathsf{h}^M)$ is an adjoint pair. The (natural) bijections
$$\mathrm{Hom}_{\mathsf{Mod}_R}(N \otimes_R M, P) \cong \mathrm{Hom}_{\mathsf{Mod}_R}(N, \mathrm{Hom}_{\mathsf{Mod}_R}(M,P))$$
induced by the adjunction
$$\mathsf{Mod}_R \underset{\mathsf{h}^M}{\overset{-\otimes_R M}{\rightleftarrows}} \mathsf{Mod}_R$$
are in fact isomorphisms of abelian groups (recall that Mod_R is abelian).

It is important to remember the following properties:

- every equivalence of categories is an adjunction,
- every right adjoint (resp. left adjoint) functor between two abelian categories is left exact (resp. right exact),
- if a functor has two left (or right) adjoints, then they are naturally isomorphic.

A.2 Yoneda Lemma

In this section we study representable functors and recall the statement of the Yoneda Lemma. More details and examples can be found, for instance, in [20].

For simplicity, all categories are assumed to be *locally small* throughout.

Let $\mathscr{C}$ be a (locally small) category. Consider the category of contravariant functors $\mathscr{C} \to \mathsf{Sets}$, i.e. the *functor category*

$$\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}).$$

For every object x of $\mathscr{C}$ there is a functor $\mathsf{h}_x \colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ defined by

$$u \mapsto \mathsf{h}_x(u) = \mathrm{Hom}_{\mathscr{C}}(u, x), \quad u \in \mathscr{C}.$$

A morphism $\phi \in \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(u, v) = \mathrm{Hom}_{\mathscr{C}}(v, u)$ gets sent to the map of sets

$$\mathsf{h}_x(\phi) \colon \mathsf{h}_x(u) \to \mathsf{h}_x(v), \quad \alpha \mapsto \alpha \circ \phi.$$

Consider the functor

$$\mathsf{h}_{\mathscr{C}} \colon \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}), \quad x \mapsto \mathsf{h}_x. \tag{A.2.1}$$

This is, indeed, a functor: for every arrow $f \colon x \to y$ in $\mathscr{C}$ and object u of $\mathscr{C}$ we can define a map of sets

$$\mathsf{h}_f\, u \colon \mathsf{h}_x(u) \to \mathsf{h}_y(u), \quad \alpha \mapsto f \circ \alpha,$$

with the property that for every morphism $\phi \colon v \to u$ in $\mathscr{C}$ there is a commutative diagram

$$\begin{array}{ccc} \mathsf{h}_x(u) & \xrightarrow{\mathsf{h}_f\, u} & \mathsf{h}_y(u) \\ {\scriptstyle \mathsf{h}_x(\phi)}\downarrow & & \downarrow{\scriptstyle \mathsf{h}_y(\phi)} \\ \mathsf{h}_x(v) & \xrightarrow{\mathsf{h}_f\, v} & \mathsf{h}_y(v) \end{array} \qquad \begin{array}{ccc} u \xrightarrow{\alpha} x & \longmapsto & u \xrightarrow{f\circ\alpha} y \\ \big\downarrow & & \big\downarrow \\ v \xrightarrow{\alpha\circ\phi} x & \longmapsto & u \xrightarrow{f\circ\alpha\circ\phi} y \end{array}$$

defining a natural transformation

$$\mathsf{h}_f \colon \mathsf{h}_x \Rightarrow \mathsf{h}_y.$$

LEMMA A.2.1 (Weak Yoneda). *The functor* $\mathsf{h}_{\mathscr{C}}$ *defined in* (A.2.1) *is fully faithful.*

Definition A.2.2 (Representable functor)**.** A functor $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ is *representable* if it lies in the essential image of $\mathsf{h}_{\mathscr{C}}$, i.e. if it is isomorphic to a functor h_x for some $x \in \mathscr{C}$. In this case, we say that the object $x \in \mathscr{C}$ represents F.

Remark A.2.3. By Lemma A.2.1, if $x \in \mathscr{C}$ represents F, then x is unique up to a unique isomorphism. Indeed, suppose we have isomorphisms

$$a\colon \mathsf{h}_x \xrightarrow{\sim} F, \quad b\colon \mathsf{h}_y \xrightarrow{\sim} F$$

in the category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$. Then there exists a unique isomorphism $x \xrightarrow{\sim} y$ inducing $b^{-1} \circ a\colon \mathsf{h}_x \xrightarrow{\sim} \mathsf{h}_y$.

Let $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ be a functor, $x \in \mathscr{C}$ an object. One can construct a map of sets

$$g_x\colon \mathrm{Hom}(\mathsf{h}_x, \mathsf{F}) \to \mathsf{F}(x). \tag{A.2.2}$$

To a natural transformation $\eta\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ one can associate the element

$$g_x(\eta) = \eta_x(\mathrm{id}_x) \in \mathsf{F}(x),$$

the image of $\mathrm{id}_x \in \mathsf{h}_x(x)$ via the map $\eta_x\colon \mathsf{h}_x(x) \to \mathsf{F}(x)$.

LEMMA A.2.4 (Strong Yoneda). *Let* $\mathsf{F} \in \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets})$ *be a functor,* $x \in \mathscr{C}$ *an object. Then the map* g_x *defined in* (A.2.2) *is bijective.*

Proof. The inverse of g_x is the map that assigns to an element $\xi \in \mathsf{F}(x)$ the natural transformation $\eta(x,\xi)\colon \mathsf{h}_x \Rightarrow \mathsf{F}$ defined as follows. For a given object $u \in \mathscr{C}$, we define

$$\eta(x,\xi)_u\colon \mathsf{h}_x(u) \to \mathsf{F}(u)$$

by sending a morphism $f\colon u \to x$ to the image of ξ under $\mathsf{F}(f)\colon \mathsf{F}(x) \to \mathsf{F}(u)$. □

Exercise A.2.5. Show that Lemma A.2.4 implies Lemma A.2.1.

Definition A.2.6 (Universal object)**.** Let $\mathsf{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. A *universal object* for F is a pair (x,ξ) where $\xi \in \mathsf{F}(x)$, such that for every pair (u,σ) with $\sigma \in \mathsf{F}(u)$, there exists a unique morphism $\alpha\colon u \to x$ with the property that $\mathsf{F}(\alpha)\colon \mathsf{F}(x) \to \mathsf{F}(u)$ sends ξ to σ.

Exercise A.2.7. Show that a pair (x,ξ) is a universal object for a functor $\mathsf{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$ if and only if the natural transformation $\eta(x,\xi)$ defined in the proof of Lemma A.2.4 is a natural isomorphism. In particular, F is representable if and only if it has a universal object.

A.3 Moduli spaces in algebraic geometry

In classical moduli theory, one is interested in the category

$$\mathscr{C} = \mathsf{Sch}_S$$

of schemes over a fixed base scheme S. Its objects are pairs (X, f), where X is a scheme and $f : X \to S$ is a morphism of schemes. Sometimes one just writes $(f : X \to S)$ to denote an object of Sch_S. A morphism $(X, f) \to (Y, g)$ in Sch_S is a morphism $p : X \to Y$ such that $g \circ p = f$. One has the following important notion in moduli theory.

Definition A.3.1 (Fine moduli space)**.** Let $\mathfrak{M}\colon \mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ be a functor. If an S-scheme $M \to S$ represents $\mathfrak{M}$, then $M \to S$ is called a fine moduli space for the moduli problem defined by $\mathfrak{M}$.

To say that $M \to S$ is a fine moduli space for a functor $\mathfrak{M}$ in particular says that $M \to S$ is unique up to unique isomorphism, and by Exercise A.2.7 it has a universal object $\xi \in \mathfrak{M}(M \to S)$ in the sense of Definition A.2.6.

Example A.3.2. The existence of fibre products in the category of schemes $\mathsf{Sch} = \mathsf{Sch}_{\operatorname{Spec}\mathbb{Z}}$ amounts to the representability of the functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ sending a scheme $A \in \mathsf{Sch}$ to the set

$$\mathrm{Hom}_{\mathsf{Sch}}(A, X) \times_{\mathrm{Hom}_{\mathsf{Sch}}(A,S)} \mathrm{Hom}_{\mathsf{Sch}}(A, Y).$$

Example A.3.3 (Global Spec)**.** Let S be a scheme, $\mathscr{A}$ a quasicoherent $\mathscr{O}_S$-algebra. Then the S-scheme $\mathrm{Spec}_{\mathscr{O}_S} \mathscr{A} \to S$ represents the functor $\mathsf{Sch}_S^{\mathrm{op}} \to \mathsf{Sets}$ sending

$$(U \xrightarrow{g} S) \;\mapsto\; \mathrm{Hom}_{\mathscr{O}_S\text{-alg}}(\mathscr{A}, g_*\mathscr{O}_U).$$

B | Commutative algebra

B.1 Frequently used theorems

LEMMA B.1.1 (Nakayama).

B.2 Tensor products

Definition B.2.1 (Tensor product of modules)**.** Let A be a ring, M and N two A-modules. The *tensor product* of M and N over A is defined to be a pair $(M \otimes_A N, p)$ where

- $M \otimes_A N$ is an A-module,
- $p\colon M \times N \to M \otimes_A N$ is a bilinear map,

such that the following universal property is satisfied: for every pair (E, q) where E is an A-module and $q\colon M \times N \to E$ is a bilinear map, there is exactly one A-linear homomorphism $\phi_q\colon M \otimes_A N \to E$ such that $q = \phi_q \circ p$.

The universal property of Definition B.2.1 can be depicted in the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad p \quad} & M \otimes_A N \\ & {\scriptstyle \forall q} \searrow \quad \swarrow {\scriptstyle \exists! \phi_q} & \\ & E & \end{array}$$

and, more importantly, can be rephrased by saying that there is a bijection

$$\mathrm{Bil}_A(M \times N, E) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Mod}_A}(M \otimes N, E), \quad q \mapsto \phi_q,$$

functorial in E.

Regarding existence of an object $(M \otimes_A N, p)$ with the required universal property, one first considers the standard basis $\{\, e_{m,n} \mid m \in M,\ n \in N \,\}$ of the direct sum $A^{\oplus M \times N}$. One then constructs the quotient module

$$M \otimes N = A^{\oplus M \times N} / T$$

where $T \subset A^{\oplus M\times N}$ is the submodule generated by elements of the form

$$\begin{aligned} &e_{m_1+m_2,n} - e_{m_1,n} - e_{m_2,n},\\ &e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2},\\ &e_{am,n} - e_{m,an},\\ &a e_{m,n} - e_{am,n}. \end{aligned}$$

The map $p\colon M\times N \to M\otimes_A N$ is defined by sending $(m,n)\mapsto [e_{m,n}]$, where the square bracket means equivalence class. One sets

$$m\otimes n = [e_{m,n}].$$

This is standard notation. Note that not all elements of $M\otimes_A N$ are of the form $m\otimes n$ for elements $m\in M$ and $n\in N$. However, every element $u\in M\otimes_A N$ can be written (non-uniquely) as a finite sum

$$u = \sum_{k=1}^{r} m_k\otimes n_k, \quad r>0.$$

Granting that the above pair $(M\otimes_A N, p)$ satisfies the universal property of Definition B.2.1 (which is an easy exercise), one has automatically that such pair is unique. Note that one has the elementary identifications

$$\begin{aligned} M\otimes_A A &= M,\\ M\otimes_A N &= N\otimes_A M,\\ (M\otimes_A N)\otimes_A P &= M\otimes_A (N\otimes_A P). \end{aligned}$$

Exercise B.2.2. If $(M_i)_{i\in I}$ is a family of A-modules, one has a canonical isomorphism

$$\bigoplus_{i\in I}(M_i\otimes_A N) \xrightarrow{\sim} \left(\bigoplus_{i\in I} M_i\right)\otimes_A N$$

for any A-module N.

Exercise B.2.3. Let A be a ring, M an A-module. Prove that the functor

$$M\otimes_A -\colon \mathsf{Mod}_A \to \mathsf{Mod}_A, \quad N\mapsto M\otimes_A N$$

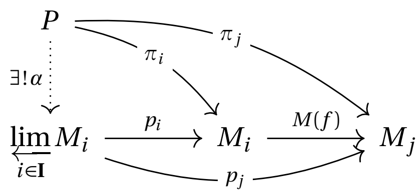
is right exact, i.e. that a surjection $N_1 \twoheadrightarrow N_2$ gets sent to a surjection $M\otimes_A N_1 \twoheadrightarrow M\otimes_A N_2$.

B.3 Universal constructions

B.3.1 Limits and colimits

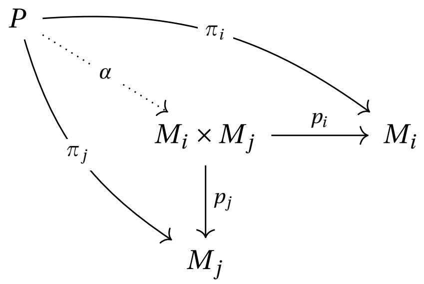
Let $\mathscr{C}$ be a category, $\mathbf{I}$ a small category. Define an $\mathbf{I}$-diagram to be just a functor $M\colon \mathbf{I}\to\mathscr{C}$. Denote by M_i the object of $\mathscr{C}$ image of the object $i\in\mathbf{I}$ via M. If $f\colon i\to j$ is an arrow in $\mathbf{I}$, the induced arrow in $\mathscr{C}$ is denoted $M(f)\colon M_i\to M_j$.

Definition B.3.1 (Limit)**.** A *limit* of an $\mathbf{I}$-diagram $M\colon \mathbf{I}\to\mathscr{C}$ is an object $\varprojlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $p_i\colon \varprojlim_{i\in\mathbf{I}} M_i \to M_i$ for every $i\in\mathbf{I}$, such that for every arrow $f\colon i\to j$ in $\mathbf{I}$ one has $p_j = M(f)\circ p_i$, and satisfying the following universal property: given an object P along with morphisms $\pi_i\colon P\to M_i$ such that $\pi_j = M(f)\circ\pi_i$ for every $f\colon i\to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha\colon P\to\varprojlim_{i\in\mathbf{I}} M_i$ such that $\pi_i = p_i\circ\alpha$ for all $i\in\mathbf{I}$.



Exercise B.3.2. The limit over the empty diagram satisfies the universal property of a final object of $\mathscr{C}$.

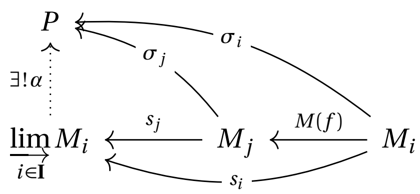
Example B.3.3 (Products are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and no morphisms between them. Then an $\mathbf{I}$-diagram $M\colon\mathbf{I}\to\mathscr{C}$ is just the choice of two objects M_i, M_j of $\mathscr{C}$. The limit of M satisfies the universal property of the product $M_i\times M_j$.



Example B.3.4 (Equalisers are limits)**.** Let $\mathbf{I}$ be the category with two objects i, j and two arrows $i\rightrightarrows j$. Then an $\mathbf{I}$-diagram $M\colon\mathbf{I}\to\mathscr{C}$ is just the choice of two parallel arrows $\phi,\psi\colon M_i\rightrightarrows M_j$ in $\mathscr{C}$. The limit of M satisfies the universal property of the equaliser of (ϕ,ψ).

Example B.3.5 (Kernels are limits)**.** This is because kernels are equalisers (in the previous example take $\psi = 0$).

Definition B.3.6 (Colimit)**.** A *colimit* of an $\mathbf{I}$-diagram $M\colon\mathbf{I}\to\mathscr{C}$ is an object $\varinjlim_{i\in\mathbf{I}} M_i$ of $\mathscr{C}$ along with an arrow $s_i\colon M_i\to\varinjlim_{i\in\mathbf{I}} M_i$ for every $i\in\mathbf{I}$, such that for every arrow $f\colon i\to j$ in $\mathbf{I}$ one has $s_i = s_j\circ M(f)$, and satisfying the following universal property: given an object P along with morphisms $\sigma_i\colon M_i\to P$ such that $\sigma_i = \sigma_j\circ M(f)$ for every $f\colon i\to j$ in $\mathbf{I}$, there exists a unique arrow $\alpha\colon\varinjlim_{i\in\mathbf{I}} M_i\to P$ such that $\sigma_i = \alpha\circ s_i$ for all $i\in\mathbf{I}$.



Exercise B.3.7. The colimit over the empty diagram satisfies the universal property of an initial object of $\mathscr{C}$ (cf. Exercise B.3.2).

Exercise B.3.8. Convince yourself that coproducts, coequalisers and cokernels are examples of colimits, along the same lines of Examples B.3.3, B.3.4 and B.3.5.

Definition B.3.9 (Filtered category)**.** A nonempty category $\mathbf{I}$ is *filtered* if for every two objects $i, j \in \mathbf{I}$ the following are true:

- there exists an object $k \in \mathbf{I}$ along with two morphisms $i \to k$ and $j \to k$, and
- for any two morphisms $f, g \in \operatorname{Hom}_{\mathbf{I}}(i, j)$ there exists an object $k \in \mathbf{I}$ along with a morphism $h \colon j \to k$ such that $h \circ f = h \circ g$ in $\operatorname{Hom}_{\mathbf{I}}(i, k)$.

The colimit of an $\mathbf{I}$-diagram $M : \mathbf{I} \to \mathscr{C}$ where $\mathbf{I}$ is a filtered category is a *filtered colimit.*

Example B.3.10. In the definition of stalk of a presheaf $\mathcal{F} \in \mathsf{pSh}(X, \mathscr{C})$ at a point $x \in X$, we have been taking

$$\begin{aligned} \mathbf{I} &= \{\, U \in \tau_X \mid x \in U \,\}^{\mathrm{op}} \\ M(U) &= \mathcal{F}(U). \end{aligned}$$

B.4 Localisation

B.4.1 General construction for modules

Let A be a ring, M an A-module. Fix a *multiplicative subset* $S \subset A$, i.e. a subset containing the identity $1 \in A$ and such that $s_1 s_2 \in S$ whenever $s_1, s_2 \in S$.

Example B.4.1. The following are key examples of multiplicative subsets:

(i) $S = \{\, f^n \mid n \geq 0 \,\}$ for some $f \in A$.

(ii) $S = A \setminus \mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal.

(iii) $S = A \setminus 0$, if A is an integral domain.

(iv) $S = A \setminus \mathcal{Z}$, where $\mathcal{Z}$ is the set of all zero-divisors in A.

Consider the equivalence relation on $M \times S$ defined by

$$\boxed{(m, s) \sim (m', s') \iff \text{there exists } u \in S \text{ such that } u(s'm - sm') = 0 \in M.}$$

We denote by m/s, or by $\frac{m}{s}$, the equivalence class of (m, s). The set of such equivalence classes

$$S^{-1}M = (M \times S)/\sim \qquad \text{(B.4.1)}$$

is an abelian group via

$$\frac{m}{s}+\frac{m'}{s'}=\frac{sm'+s'm}{ss'},$$

and if $M=A$ then $S^{-1}A$ becomes a ring via

$$\frac{a}{s}\cdot\frac{a'}{s'}=\frac{aa'}{ss'}.$$

The $\mathbb{Z}$-module $S^{-1}M$ is an $S^{-1}A$-module via

$$\frac{a}{s}\cdot\frac{m}{s'}=\frac{am}{ss'}. \tag{B.4.2}$$

Here 'am' refers to the A-module structure on M.

Definition B.4.2 (Localisation of a module)**.** The localisation of M with respect to S is the $S^{-1}A$-module $S^{-1}M$, where the linear structure is given by Equation (B.4.2).

Localisation is functorial: if $\phi: N\to M$ is an A-linear map, there is an induced map

$$S^{-1}\phi: S^{-1}N \longrightarrow S^{-1}M, \qquad \frac{n}{s}\mapsto\frac{\phi(n)}{s}.$$

This map is $S^{-1}A$-linear, indeed if $a/t\in S^{-1}A$ then

$$S^{-1}\phi\left(\frac{a}{t}\cdot\frac{n}{s}\right)=S^{-1}\phi\left(\frac{an}{ts}\right)=\frac{\phi(an)}{ts}=\frac{a\cdot\phi(n)}{ts}=\frac{a}{t}\cdot\frac{\phi(n)}{s}=\frac{a}{t}\cdot S^{-1}\phi\left(\frac{n}{s}\right).$$

Remark B.4.3. If $0\in S$, then $S^{-1}M=0$.

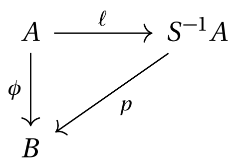
Notation B.4.4. If $S=\{f^n\mid n\geq 0\}$ as in Example B.4.1 (i) above, then we write M_f for the localisation. If $S=A\setminus\mathfrak{p}$ as in Example B.4.1 (ii) above, then we write $M_{\mathfrak{p}}$ for the localisation. Do not confuse M_f and $M_{(f)}$ when $(f)=fA\subset A$ is a prime ideal!

B.4.2 Localisation of a ring and its universal property

Set $M=A$. There is a canonical ring homomorphism

$$\ell: A\to S^{-1}A, \quad a\mapsto\frac{a}{1}$$

sending S inside the group of invertible elements of $S^{-1}A$ (the inverse of $s/1$ being $1/s$), and making the pair $(S^{-1}A,\ell)$ universal with this property: whenever one has a ring homomorphism $\phi: A\to B$ such that $\phi(S)\subset B^\times$, there is exactly one ring homomorphism $p: S^{-1}A\to B$ such that $\phi=p\circ\ell$.



Explicitly, the map p is defined by $p(a/s)=\phi(a)\phi(s)^{-1}$.

Remark B.4.5. The localisations of the form A_f are crucial in algebraic geometry. In A_f, the equivalence relation defining the localisation reads

$$\frac{a}{f^n} = \frac{b}{f^m} \Longleftrightarrow \text{there exists } k \geq 0 \text{ such that } f^k(af^m - bf^n) = 0 \in A.$$

In particular, one has that $A_f = 0$ if and only if f is nilpotent, and

$$A_f \ni 0 = \frac{0}{1} = \frac{a}{f^n} \quad \Longleftrightarrow \quad \text{there exists } k \geq 0 \text{ such that } f^k a = 0 \in A.$$

The following lemma is of key importance to us.

Lemma B.4.6. *Let A be a ring, and $\ell\colon A \to S^{-1}A$ a localisation. Sending $\mathfrak{r} \mapsto \ell^{-1}(\mathfrak{r})$ establishes a bijection*

$$\begin{array}{ccc} \{\text{prime ideals } \mathfrak{r} \subset S^{-1}A\} & \xrightarrow{\ \simeq\ } & \{\text{prime ideals } \mathfrak{q} \subset A \text{ such that } \mathfrak{q} \cap S = \emptyset\} \\ \| & & \downarrow \\ \operatorname{Spec} S^{-1}A & & \operatorname{Spec} A \end{array}$$

having as inverse the extension operation, sending

$$\mathfrak{q} \mapsto \mathfrak{q} \cdot S^{-1}A = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{q},\, f \in S \right\} \subset S^{-1}A.$$

Corollary B.4.7. *For any prime ideal $\mathfrak{p} \subset A$ the ring*

$$A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in A,\, f \notin \mathfrak{p} \right\}$$

is local, with maximal ideal

$$\mathfrak{p} \cdot A_{\mathfrak{p}} = \left\{ \frac{a}{f} \,\middle|\, a \in \mathfrak{p},\, f \notin \mathfrak{p} \right\} \subset A_{\mathfrak{p}}.$$

Proof. Indeed, the correspondence of Lemma B.4.6 becomes, in the case $S = A \setminus \mathfrak{p}$,

$$\{\text{prime ideals } \mathfrak{r} \subset A_{\mathfrak{p}}\} \xrightarrow{\ \simeq\ } \{\text{prime ideals } \mathfrak{q} \subset A \text{ such that } \mathfrak{q} \subset \mathfrak{p}\}$$

and since its inverse (extension along $A \to A_{\mathfrak{p}}$) is inclusion-preserving it follows that every prime ideal $\mathfrak{r} \subset A_{\mathfrak{p}}$ must be contained in $\mathfrak{p} \cdot A_{\mathfrak{p}}$. This means that $\mathfrak{p} \cdot A_{\mathfrak{p}}$ is the unique maximal ideal. □

Exercise B.4.8. If $(A, \mathfrak{m})$ is a local ring, then $A = A_{\mathfrak{m}}$.

Warning B.4.9. In the case when B is a graded ring and $\mathfrak{p}$ is a homogeneous prime ideal, we use the notation $B_{\mathfrak{p}}$ for the localisation of B at the multiplicative subset consisting of *homogeneous* elements that are not in $\mathfrak{p}$.

PROPOSITION B.4.10 ([12, Prop. 5.8]). *If $\mathfrak{m} \subset A$ is a maximal ideal and $k > 0$ is an integer, there is a natural ring isomorphism*

$$A/\mathfrak{m}^k \xrightarrow{\sim} A_{\mathfrak{m}}/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k.$$

It induces isomorphisms

$$\mathfrak{m}^h/\mathfrak{m}^k \xrightarrow{\sim} (\mathfrak{m} \cdot A_{\mathfrak{m}})^h/(\mathfrak{m} \cdot A_{\mathfrak{m}})^k$$

for every $h \leq k$.

LEMMA B.4.11. *Let A be a ring, $S \subset A$ a multiplicative subset. Then $\ell \colon A \to S^{-1}A$ is injective if and only if S contains no zero divisors.*

Proof. Suppose $a/1 = 0/1$ in $S^{-1}A$. Then there is $u \in S$ such that $au = 0$. But u is not a zero divisor, thus $a = 0$. □

Example B.4.12. Let A be an integral domain, which means that $(0) \subset A$ is prime. Then the localisation

$$A_{(0)} = \left\{ \frac{a}{b} \,\middle|\, a \in A,\, b \in A \setminus 0 \right\}$$

is a field, called the *fraction field* of A, that we denote by $\mathrm{Frac}(A)$. The canonical map $\ell \colon A \to \mathrm{Frac}(A)$ is injective by Lemma B.4.11.

Example B.4.13. Let A be a ring. Consider $S = A \setminus \mathcal{Z}$ as in Example B.4.1 (iv). The localisation $S^{-1}A$ is called the *total ring of fractions* of A. By Lemma B.4.11, $S = A \setminus \mathcal{Z}$ is the largest multiplicative set such that $\ell \colon A \to S^{-1}A$ is injective.

Example B.4.14. Let $A = \mathbb{Z}$. Fix a prime number $p \in \mathbb{Z}$. Then the localisation map

$$\mathbb{Z} \to \mathbb{Z}_{(p)} = \left\{ \frac{n}{m} \,\middle|\, n \in \mathbb{Z},\, p \nmid m \right\}$$

is injective, and so is the localisation map

$$\mathbb{Z} \to \mathbb{Z}_p = \left\{ \frac{n}{p^k} \,\middle|\, n \in \mathbb{Z},\, k \geq 0 \right\}.$$

LEMMA B.4.15. *If A is reduced and $S \subset A$ is a multiplicative subset, then $S^{-1}A$ is also reduced.*

Proof. Assume there exists $a \in A$, $s \in S$ and $r \in \mathbb{Z}_{>0}$ such that $0/1 = (a/s)^r = a^r/s^r \in S^{-1}A$. Then there exists $u \in S$ such that $ua^r = 0 \in A$, thus $(ua)^r = 0$, and hence $ua = 0$ by assumption. But this means $0/1 = a/1 \in S^{-1}A$, and thus $0/1 = (a/1)(1/s) = a/s \in S^{-1}A$. □

B.4.3 Exactness of localisation

LEMMA B.4.16. *Let A be a ring, $S \subset A$ a multiplicative subset, M an A-module. Then, there is a canonical isomorphism of $S^{-1}A$-modules*

$$\phi : S^{-1}M \xrightarrow{\sim} M \otimes_A S^{-1}A.$$

Proof. First of all, the $S^{-1}A$-module structure on $M \otimes_A S^{-1}A$ si defined (on generators) by

$$\frac{a}{t} \cdot \left(m \otimes \frac{b}{s}\right) = m \otimes \frac{ab}{ts}.$$

The map ϕ is defined by

$$\phi\left(\frac{m}{s}\right) = m \otimes \frac{1}{s}.$$

It is $S^{-1}A$-linear, since

$$\begin{aligned}
\phi\left(\frac{a}{t} \cdot \frac{m}{s}\right) &= \phi\left(\frac{am}{ts}\right) \\
&= am \otimes \frac{1}{ts} \\
&= m \otimes \frac{a}{ts} \\
&= \frac{a}{t} \cdot \left(m \otimes \frac{1}{s}\right) \\
&= \frac{a}{t} \cdot \phi\left(\frac{m}{s}\right).
\end{aligned}$$

Its inverse is given by $m \otimes (a/s) \mapsto (am)/s$. □

PROPOSITION B.4.17. *Let A be a ring, $S \subset A$ a multiplicative subset. Then, sending $M \mapsto S^{-1}M$ defines an exact functor from A-modules to $S^{-1}A$-modules.*

Proof. Fix a short exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} N \xrightarrow{\pi} P \longrightarrow 0$$

of A-modules. We already know that

$$S^{-1}M \xrightarrow{S^{-1}\iota} S^{-1}N \xrightarrow{S^{-1}\pi} S^{-1}P \longrightarrow 0$$

is exact, since this sequence is isomorphic to

$$M \otimes_A S^{-1}A \longrightarrow N \otimes_A S^{-1}A \longrightarrow P \otimes_A S^{-1}A \longrightarrow 0$$

by Lemma B.4.16, and tensor product (by any A-module, e.g. $S^{-1}A$) is a right exact functor. So we only need to show that

$$S^{-1}M \xrightarrow{S^{-1}\iota} S^{-1}N$$

is injective. Assume there is an element $m/s \in S^{-1}M$ such that $0 = 0/1 = S^{-1}\iota(m/s) = \iota(m)/s \in S^{-1}N$. Then there exists $u \in S$ such that $0 = u\iota(m) = \iota(um)$ in N. This implies $um = 0 \in M$, hence $m/s = um/us = 0/us = 0$. □

B.5 Normalisation

Normal schemes are either regular or ‘mildly singular’ schemes. For instance, a key property of normal schemes is that singularities only occur in codimension 2 or higher. We now give precise definitions.

Definition B.5.1 (Normality)**.** We say that

(i) an integral domain A is *normal* if it is integrally closed in $\operatorname{Frac} A$,

(ii) a ring is *normal* if all its local rings are normal domains,

(iii) a scheme is *normal* if $\mathscr{O}_{X,x}$ is a normal integral domain for every $x \in X$.

Remark B.5.2. A scheme is normal if and only if it is ‘locally normal’ in the sense of **??**. The terminology ‘locally normal’ is never used though.

Remark B.5.3. By definition, a ring A is normal precisely when $\operatorname{Spec} A$ is a normal scheme.

Example B.5.4. A regular scheme is normal. A normal scheme is reduced. To see the latter, it is enough to observe that for any open subset $U \subset X$ there is an injective ring homomorphism

$$\mathscr{O}_X(U) \hookrightarrow \prod_{x \in U} \mathscr{O}_{X,x}$$

since $\mathscr{O}_X$ is a sheaf (cf. Lemma 2.4.1), where $\mathscr{O}_{X,x}$ is reduced for every $x \in X$, since it is a domain.

Example B.5.5 (Locally factorial schemes are normal)**.** A scheme is *locally factorial* if $\mathscr{O}_{X,x}$ is a UFD for every $x \in X$. A UFD is normal, so a locally factorial scheme is normal.

Example B.5.6. Let A be a normal domain. Then $S^{-1}A$ is a normal domain for any multiplicative subset $S \subset A$ (see **??** for a proof). Thus, $\operatorname{Spec} A$ is normal, and so is any principal open $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$.

Caution B.5.7. It is not true that if $\operatorname{Spec} A$ is normal, then A is an integral domain: for instance, if $\mathbb{F}$ is a field, then

$$\operatorname{Spec} \mathbb{F} \amalg \operatorname{Spec} \mathbb{F} = \operatorname{Spec} \mathbb{F} \times \mathbb{F} = \operatorname{Spec} \mathbb{F}[x]/(x(x-1))$$

is a normal scheme, but $\mathbb{F}[x]/(x(x-1))$ is not a domain.

Exercise B.5.8. Show that $\operatorname{Spec} \mathbb{C}[x,y,z]/(x^2+y^2-z^2)$ is normal but not locally factorial.

Exercise B.5.9. Let $\mathbb{F}$ be a field, with $\operatorname{char} \mathbb{F} \neq 2$. Show that the following schemes are normal.

- $\operatorname{Spec}\mathbb{Z}[x]/(x^2-n)$, where $n\in\mathbb{Z}$ is square-free and congruent to 3 modulo 4.
- $\operatorname{Spec}\mathbb{F}[x_1,\dots,x_n]/(x_1^2+\cdots+x_m^2)$, where $n\geq m\geq 3$.
- $\operatorname{Spec}\mathbb{F}[x,y,z,w]/(xy-zw)$.

PROPOSITION B.5.10. *Let X be a scheme.*

(A) *The following conditions are equivalent:*

(1) *X is normal.*

(2) *$\mathscr{O}_X(U)$ is a normal ring for every affine open $U\subset X$.*

(3) *There is an affine open covering $X=\bigcup_{i\in I}U_i$ such that $\mathscr{O}_X(U_i)$ is a normal ring for every $i\in I$.*

(4) *There is an open covering $X=\bigcup_{j\in J}V_j$ such that V_j is normal for every $j\in I$.*

Moreover, every open subscheme of a normal scheme X is normal.

(B) *If X is quasicompact, the above conditions are equivalent to*

(5) *$\mathscr{O}_{X,x}$ is a normal domain for every closed point $x\in X$.*

(C) *If X is integral, the above conditions are equivalent to*

(6) *$\mathscr{O}_X(U)$ is a normal domain for every affine open $U\subset X$.*

Proof. To prove (A), combine the Locality Lemma (cf. **??**), Remark B.5.2 and **??** with one another.

To prove (B), argue as in the proof of **??**.

To prove (C), it is enough to use the definition of integral scheme (cf. **??**) and point (A). □

Remark B.5.11. By the above proof, the first two conditions are equivalent even without assuming quasicompactness.

LEMMA B.5.12 ([11, Ch. 4, Lemma 1.13]). *Let A be a normal noetherian ring of dimension at least 1. Then*

$$A=\bigcap_{\substack{\mathfrak{p}\in\operatorname{Spec}A\\ \operatorname{ht}(\mathfrak{p})=1}}A_{\mathfrak{p}},$$

the intersection being taken inside $\operatorname{Frac}A$*.*

COROLLARY B.5.13. *Let X be a normal locally noetherian scheme. Let $Z\subset X$ be a closed subset of codimension at least 2. Then the natural map*

$$\mathscr{O}_X(X)\to\mathscr{O}_X(X\setminus Z)$$

is an isomorphism.

Proof. □

Example B.5.14. Note that Corollary B.5.13 reproves the content of Example 3.1.73, i.e. the identity

$$\mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}}) = \mathscr{O}_{\mathbb{A}^n_{\mathbf{k}}}(\mathbb{A}^n_{\mathbf{k}} \setminus \{0\})$$

for any $n \geq 2$.

There is a procedure, called *normalisation*, which does the following. Given, as input, an integral scheme X, one constructs a pair $(\widetilde{X}, \pi)$ where $\widetilde{X}$ is a normal scheme and $\pi \colon \widetilde{X} \to X$ is a morphism of schemes which is universal in the following sense: for every pair (Y, f) where Y is a normal scheme and $f: Y \to X$ is normal, there exists exactly one morphism $\alpha_f : Y \to \widetilde{X}$ such that $\pi \circ \alpha_f = f$.

Remark B.5.15. The normalisation of an integral scheme, if it exists (which it does, see Theorem B.5.17 below), is unique up to unique isomorphism.[1] Moreover, the universal property also shows that if $\pi \colon \widetilde{X} \to X$ is the normalisation and $U \subset X$ is open, then the base change map $\pi^{-1}(U) \to U$ is the normalisation of U.

In the affine case, the normalisation is easy to construct, as the following lemma shows.

LEMMA B.5.16. *Let A be an integral domain. Let $\widetilde{A} \subset \operatorname{Frac} A$ be the integral closure of A. Then the morphism*

$$\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$$

induced by the inclusion $A \hookrightarrow \widetilde{A}$ is the normalisation of $\operatorname{Spec} A$.

Proof. □

THEOREM B.5.17. *Let X be an integral scheme. Then there exists a (unique) normalisation $(\widetilde{X}, \pi)$. If X is an integral algebraic $\mathbf{k}$-variety, then the normalisation morphism $\pi \colon \widetilde{X} \to X$ is finite; in particular, $\widetilde{X}$ is an algebraic $\mathbf{k}$-variety.*

Proof. □

PROPOSITION B.5.18 ([11, Ch. 4, Cor. 1.30]). *Let X be an integral algebraic variety. Then the set of points $x \in X$ such that $\mathscr{O}_{X,x}$ is normal is open.*

Example B.5.19 (Nodal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^2(x+1))$. Then A is not normal. Let us determine its normalisation.

Example B.5.20 (Cuspidal cubic)**.** Let $A = \mathbf{k}[x, y]/(y^2 - x^3)$. Then A is not normal. Let us determine its normalisation.

[1] The normalisation being defined as a *pair*, by isomorphism we mean an isomorphism in the category Sch_X of X-schemes.

B.6 Embedded components

On a locally noetherian scheme X there are a bunch of points that are more relevant than all other points, in the sense that they reveal part of the behaviour of the structure sheaf: these points are the *associated points* of X. Some of these points are already familiar: they are the generic points, i.e. the points corresponding to the irreducible components of X. The other associated points correspond to the so-called *embedded components* of X. If X is reduced, it has no embedded components.

Let R be a commutative ring with unity, and let M be an R-module. If $m \in M$, we let

$$\operatorname{Ann}_R(m) = \{ r \in R \mid r \cdot m = 0 \} \subset R$$

denote its annihilator. A prime ideal $\mathfrak{p} \subset R$ is said to be *associated to* M if $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$. The set of all associated primes is denoted

$$\mathsf{Ass}_R(M) = \{ \mathfrak{p} \mid \mathfrak{p} \text{ is associated to } M \}.$$

LEMMA B.6.1. *Let* $\mathfrak{p} \subset R$ *be a prime ideal. Then* $\mathfrak{p} \in \mathsf{Ass}_R(M)$ *if and only if* $R/\mathfrak{p}$ *is an* R*-submodule of* M*.*

Proof. If $\mathfrak{p} = \operatorname{Ann}_R(m)$ for some $m \in M$, consider the map $\phi_m \colon R \to M$ defined by $\phi_m(r) = r \cdot m$. Since its kernel is by definition $\operatorname{Ann}_R(m)$, the quotient $R/\mathfrak{p}$ is an R-submodule of M. Conversely, given an R-linear inclusion $i \colon R/\mathfrak{p} \hookrightarrow M$, consider the composition $\phi \colon R \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$. Then $\phi_{i(1)}(r) = r \cdot i(1) = i(r + \mathfrak{p}) = \phi(r)$ for all $r \in R$, i.e. $\phi = \phi_{i(1)}$. □

Note that if $\mathfrak{p} \in \mathsf{Ass}_R(M)$ then $\mathfrak{p}$ contains the annihilator of M, i.e. the ideal

$$\operatorname{Ann}_R(M) = \{ r \in R \mid r \cdot m = 0 \text{ for all } m \in M \} \subset R.$$

Definition B.6.2 (Isolated primes)**.** The minimal elements (with respect to inclusion) in the set

$$\{ \mathfrak{p} \subset R \mid \mathfrak{p} \supset \operatorname{Ann}_R(M) \}$$

are called *isolated primes* of M.

From now on we assume R is noetherian and $M \neq 0$ is finitely generated. We have the following result.

THEOREM B.6.3 ([17, Theorem 5.5.10 (a)]). *Let* R *be a noetherian ring,* $M \neq 0$ *a finitely generated* R*-module. Then* $\mathsf{Ass}_R(M)$ *is a finite nonempty set containing all isolated primes.*

Definition B.6.4 (Embedded primes)**.** The non-isolated primes in $\mathsf{Ass}_R(M)$ are called the *embedded primes* of M.

Moreover, we have the following facts:

- the R-module M has a *composition series*, i.e. a filtration by R-submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s = M$$

such that $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. This series is not unique. However, for a prime ideal $\mathfrak{p} \subset R$, the number of times it occurs among the $\mathfrak{p}_i$ does not depend on the composition series. These primes are precisely the elements of $\mathsf{Ass}_R(M)$.

- Any ideal $I \subset R$ has a *primary decomposition*, i.e. an expression as intersection

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

of primary ideals. A proper ideal $\mathfrak{q} \subsetneq R$ is called *primary* if whenever a product xy lies in $\mathfrak{q}$, either x or a power of y lies in $\mathfrak{q}$. Put differently, every zero-divisor in $R/\mathfrak{q}$ is nilpotent. One verifies that the radical of a primary ideal is prime, and one says that $\mathfrak{q}$ is $\mathfrak{p}$-primary if $\sqrt{\mathfrak{q}} = \mathfrak{p}$. One can always ensure that the decomposition is irredundant, i.e. removing any $\mathfrak{q}_i$ changes the intersection, and $\sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$ for all $i \neq j$.

Exercise B.6.5. Let $I \subset R$ be an ideal. Show that the set

$$\{\sqrt{\mathfrak{q}_i}\}_i$$

is determined by I. Then show that elements of $\mathsf{Ass}_R(R/I)$ are precisely the radicals of the primary ideals in a primary decomposition of I. In symbols,

$$\mathsf{Ass}_R(R/I) = \{\sqrt{\mathfrak{q}_i}\}_i.$$

Exercise B.6.6. Let $R = \mathbf{k}[x,y]$, $I = (xy, y^2)$ and $M = R/I$. Show that $\mathsf{Ass}_R(M) = \{(y),(x,y)\}$.

The most boring situation is when R is an integral domain, in which case the generic point $\xi \in \operatorname{Spec} R$ is the only associated (and clearly isolated) prime. More generally, a reduced affine scheme $\operatorname{Spec} R$ has no embedded primes (in particular no embedded points, see below), i.e. the only associated primes are the isolated (minimal) ones, corresponding to its irreducible components.

Let R be an integral domain. For an ideal $I \subset R$, one often calls the associated primes of I the associated primes of R/I. The minimal primes above $I = \operatorname{Ann}_R(R/I)$ (i.e. containing I) correspond to the irreducible components of the closed subscheme

$$\operatorname{Spec} R/I \subset \operatorname{Spec} R,$$

whereas for every embedded prime $\mathfrak{p} \subset R$ there exists a minimal prime $\mathfrak{p}'$ such that $\mathfrak{p}' \subset \mathfrak{p}$. Thus $\mathfrak{p}$ determines an *embedded component* — a subvariety $\mathrm{V}(\mathfrak{p})$ embedded in an irreducible component $\mathrm{V}(\mathfrak{p}')$. If the embedded prime $\mathfrak{p}$ is maximal, we talk about an *embedded point*.

Fact B.6.7. An algebraic curve (an algebraic variety of dimension 1) has no embedded points if and only if it is Cohen–Macaulay (the formal definition is given in **??**). However, there can be nonreduced Cohen–Macaulay curves: those curves with a fat component, such as the affine plane curve $\operatorname{Spec} \mathbf{k}[x, y]/x^2 \subset \mathbb{A}^2$. These objects often have moduli, i.e. deform (even quite mysteriously) in positive dimensional families. See [3, 4, 18, 19] for generalities on multiple structures on schemes.

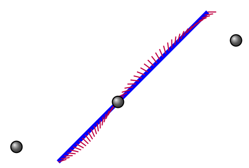


Figure B.1: A thickened (Cohen–Macaulay) curve with an embedded point and two isolated (possibly fat) points.

Remark B.6.8. An embedded component $\mathrm{V}(\mathfrak{p})$, where $\mathfrak{p}$ is the radical of some primary ideal $\mathfrak{q}$ appearing in a primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_e$, is of course embedded in some irreducible component $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$, but $\mathrm{V}(\mathfrak{q})$ is not a *subscheme* of $\mathrm{V}(\mathfrak{p}')$, because the fuzziness caused by nilpotent behavior (i.e. the difference between $\mathfrak{q}$ and its radical $\mathfrak{p}$) makes the bigger scheme $\mathrm{V}(\mathfrak{q}) \supset \mathrm{V}(\mathfrak{p})$ 'stick out' of $\mathrm{V}(\mathfrak{p}') \subset \operatorname{Spec} R/I$.

Example B.6.9. Consider $R = \mathbf{k}[x, y]$ and $I = (xy, y^2)$. A primary decomposition of I is

$$I = (x, y)^2 \cap (y).$$

However, $\operatorname{Spec} R/(x, y)^2$ is not scheme-theoretically contained in $\operatorname{Spec} R/y$.

In general, a subscheme Z of scheme Y has an embedded component if there exists a dense open subset $U \subset Y$ such that $Z \cap U$ is dense in Z but the scheme-theoretic closure of $Z \cap U \subset Z$ does not equal Z scheme-theoretically. For instance, if Y is irreducible, we say that $p \in Y$ supports an embedded point of a closed subscheme $Z \subset Y$ if $\overline{Z \cap (Y \setminus p)} \neq Z$ as schemes. In the example above, where $Y = \mathbb{A}^2$ and $Z = \operatorname{Spec} \mathbf{k}[x, y]/(xy, y^2)$, the scheme-theoretic closure of $Z \cap (\mathbb{A}^2 \setminus 0) \subset Z$ is not equal to Z.

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Andrea T. Ricolfi
SISSA, Via Bonomea 265, 34136 (Italy)
aricolfi@sissa.it