ALGEBRAIC GEOMETRY — WARM-UP QUESTIONS

This is a list of warm-up questions. You should be able to argue with examples and counterexamples, know what causes a pathology and how to fix it. Some questions lack complete assumptions: you should figure them out by yourself. Check how many you can answer without looking at the notes. Then repeat until you reach 50.

Question 1. Provide examples of:

(a) a bijective closed immersion which is not an isomorphism (besides $X_{\mathrm{red}} \to X$)
(b) a bijective morphism which is not an isomorphism (besides $X_{\mathrm{red}} \to X$ and (a))
(c) an everywhere nonreduced scheme
(d) a regular $\mathbb{F}$-scheme which is not smooth over $\mathbb{F}$
(e) a bijective morphism which is not flat nor a closed immersion
(f) a rational function on $\mathbb{P}^1_{\mathbb{C}}$ undefined at $(0:1)$
(g) a morphism with two fibres of different dimension
(h) a flat ramified morphism
(i) an unramified nonflat morphism
(j) an irreducible (resp. reduced) $\mathbb{F}$-scheme X such that $X_{\overline{\mathbb{F}}}$ is not irreducible (resp. reduced)
(k) a nonseparated presheaf satisfying the gluing axiom
(l) a morphism with exactly one nonreduced fibre
(m) an impure connected subvariety of $\mathbb{A}^3_{\mathbf{k}}$
(n) a birational map $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{A}^2_{\mathbf{k}}$. Is there one that can be extended to a morphism?
(o) a birational map $\mathbb{P}^2_{\mathbf{k}} \dashrightarrow \mathbb{P}^1_{\mathbf{k}} \times_{\mathbf{k}} \mathbb{P}^1_{\mathbf{k}}$
(p) a union of affine schemes which is not affine
(q) a quasiseparated nonseparated morphism, and a non-quasiseparated morphism
(r) non-functoriality of Proj, and non-isomorphic A-algebras with isomorphic Proj
(s) a scheme which is nonreduced at exactly two points of your choice
(t) a non-noetherian scheme all of whose local rings are noetherian
(u) a $\mathbf{k}$-scheme not locally of finite type
(v) a rational singular affine surface, a rational singular affine 3-fold
(w) the Jacobian criterion in action
(x) a morphism where '>' holds in the theorem on dimension of fibres.
(y) an étale morphism (not an isomorphism) between 2-dimensional schemes
(z) a morphism which is not étale at exactly one point

Question 2. Can an affine variety contain a proper subvariety of positive dimension?

Question 3. What makes $\operatorname{Spec}\mathbb{Z}$ special in the category of schemes?

Question 4. Let X be a scheme, $x \in X$. How do you construct $\operatorname{Spec}\mathscr{O}_{X,x} \to X$? Is it flat?

Question 5. Can you construct the rational parametrisation of the nodal cubic $Y \subset \mathbb{A}^2$?

Question 6. Is every scheme with finitely many points 0-dimensional?

Question 7. Let X be a proper integral $\mathbb{F}$-variety. Can you prove that $\mathscr{O}_X(X) = \mathbb{F}$?

Question 8. Why do cokernel and inverse image need sheafification?

Question 9. Can you prove that closed immersions are local on the target?

Question 10. Can you prove that integral = reduced and irreducible?

Question 11. Can you prove that projective over affine implies proper?

Question 12. Can you prove that an affine projective $\mathbb{F}$-scheme is finite?

Question 13. Can you describe the correspondence between closed subschemes and ideal sheaves?

Question 14. Why is the Zariski topology almost never Hausdorff?

Question 15. Is every open subset of an affine scheme quasicompact?

Question 16. Is every affine open of $\mathbb{A}^n$ principal? What about $\mathbb{P}^n$?

Question 17. Is $\mathbb{A}^4 \setminus 0 \subset \mathbb{A}^4$ principal?

Question 18. Why is $\mathbb{A}^n$ not proper if $n > 0$?

Question 19. What happens when you project the twisted cubic $C \subset \mathbb{P}^3$ on $\mathbb{P}^2$?

Question 20. If $\operatorname{Spec} A$ is irreducible, is A a domain?

Question 21. Can you prove that an irreducible scheme has a unique generic point?

Question 22. Let R, S be rings. Is $\operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec} R, \operatorname{Spec} S)$ nonempty?

Question 23. Can you construct the structure sheaf of $\operatorname{Spec} A$?

Question 24. Can you describe morphisms into an affine scheme?

Question 25. How to check that $f : X \to Y$ is dominant looking at $\mathscr{O}_Y \to f_* \mathscr{O}_X$?

Question 26. Is a quasiprojective $\mathbb{F}$-variety separated over $\mathbb{F}$?

Question 27. How do you construct the d-th Veronese map of $\mathbb{P}^n$?

Question 28. How do you make a locally closed subset Z of a scheme X into a reduced locally closed subscheme?

Question 29. What is meant by 'dimension is local'?

Question 30. Explain at least two results which rely on Noether normalisation.

Question 31. When is the dim-codim formula true? How easily can it fail?

Question 32. Can you describe the points of a fibre product of schemes?

Question 33. What can be said about affine base change of a projective scheme?

Question 34. What is the scheme-theoretic fibre of a morphism? Can you compute it in a few examples? Can you describe its points and their local rings?

Question 35. Do an infinitesimal study of the morphism $\operatorname{Spec} \mathbf{k}[x, y]/y - x^2 \to \operatorname{Spec} \mathbf{k}[y]$.

Question 36. Is every open immersion quasicompact? What about closed immersions?

Question 37. Can you characterise flat closed immersions?

Question 38. What is the relation between closed points and rational points?

Question 39. Why are affine scheme separated? Why is $\mathbb{P}^n_{\mathbb{Z}}$ separated?

Question 40. If $\operatorname{Bl}_0 \mathbb{A}^2 \to \mathbb{A}^2$ flat? Is it unramified? Is $\operatorname{Bl}_0 \mathbb{A}^2$ affine? Is it projective?

Question 41. How to describe the tangent space via dual numbers? Can you define the tangent map to a morphism and describe it using this characerisation?

Question 42. Is every flat morphism open?

Question 43. List all the properties you know of the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ sending $t \mapsto t^n$. Same for $\operatorname{Spec} \mathbf{k}[x, y]/y - x^2 \to \operatorname{Spec} \mathbf{k}[y]$ and $\mathbb{A}^1 \to \operatorname{Spec} \mathbb{C}[x, y]/y^2 - x^3$.

Question 44. Describe flatness over a regular curve.

Question 45. Can the fibres of a morphism between projective varieties be affine?

Question 46. What is the affine local model of the twisted cubic?

Question 47. Is $\operatorname{Spec}\mathbb{C}[x_i \mid i \in \mathbb{N}]/(x_i^2 \mid i \in \mathbb{N})$ irreducible? is it locally noetherian? is it finite-dimensional? What is its reduction?

Question 48. Is $\operatorname{Spec}\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ connected? finite-dimensional? reduced? locally noetherian?

Question 49. Let X and Y be reduced $\mathbb{F}$-varieties. Is $X \times_{\mathbb{F}} Y$ reduced?

Question 50. How would you define a scheme structure on the singular locus $X_{\mathsf{sing}} \subset X$ of a variety X?

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