

ALGEBRAIC GEOMETRY 2026/27 — EXERCISE SHEET #1

Exercise 1. Find a presheaf which satisfies the gluing axiom but not the separation axiom.

Exercise 2. Prove that the presheaf of bounded continuous functions on $X = \mathbb{R}$ is separated but does not satisfy the gluing axiom. What is its sheafification?

Exercise 3. Let A be a ring. Let $\eta: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves of A -modules. Show that $U \mapsto \ker \eta_U$ and $U \mapsto \operatorname{coker} \eta_U$ are presheaves, and satisfy the universal property of kernel and cokernel, respectively, in the category of presheaves.

Exercise 4. Let $\eta_1, \eta_2: \mathcal{F} \rightarrow \mathcal{G}$ be morphisms of presheaves, where $\mathcal{G}$ is a sheaf. Show that if $\eta_{1,x} = \eta_{2,x}$ for every $x \in X$, then $\eta_1 = \eta_2$.

Exercise 5. Let X be a topological space, x a point, G a nontrivial abelian group. Find a necessary and sufficient (topological) condition for the constant sheaf $\underline{G}_X$ to agree with the skyscraper sheaf G_x .

Exercise 6. Let A be a ring. For a nonempty open subset U of a topological space X , consider the functor $\Gamma(U, -): \operatorname{Sh}(X, \operatorname{Mod}_A) \rightarrow \operatorname{Mod}_A$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$. Show that it is left exact. That is, it transforms an exact sequence of sheaves $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H}$ into an exact sequence of A -modules

$$0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U).$$

Exercise 7. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous maps of topological spaces. Show that there is a canonical natural isomorphism

$$f^{-1} \circ g^{-1} \xrightarrow{\sim} (g \circ f)^{-1}$$

of functors $\operatorname{Sh}(Z, \mathcal{C}) \rightarrow \operatorname{Sh}(X, \mathcal{C})$.

Exercise 8. Show that if $f: X \hookrightarrow Y$ is the inclusion of a subspace, then the counit

$$f^{-1} f_* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

is an isomorphism for every $\mathcal{F} \in \operatorname{Sh}(X, \mathcal{C})$. (**Hint:** check it on stalks).

Exercise 9. Consider $\mathcal{O}_X$ -modules $\mathcal{F}$ and $\mathcal{G}$. Form the Hom-sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. Show the following:

- (i) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}_X$ -modules.
- (ii) $\mathcal{H}om_{\mathcal{O}_X}(-, -)$ is functorial in both variables.
- (iii) As functors $\operatorname{Mod}(\mathcal{O}_X) \rightarrow \operatorname{Mod}(\mathcal{O}_X)$, we have that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -)$ is (covariant and) left exact, and $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G})$ is (contravariant and) left exact.