

ALGEBRAIC GEOMETRY 2026/27 — EXERCISE SHEET #2

Exercise 1. Let A be a ring, $\mathfrak{p} \subset A$ a prime ideal. Set $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. Show that $\text{Frac } A/\mathfrak{p} = \kappa(\mathfrak{p})$. In particular, if $\mathfrak{m} \subset A$ is maximal, then $\kappa(\mathfrak{m}) = A/\mathfrak{m}$.

Exercise 2. Show that a morphism of schemes $f: X \rightarrow Y$ is quasicompact if and only if Y has an affine open cover $Y = \bigcup_{i \in I} Y_i$ such that $f^{-1}(Y_i)$ is quasicompact for all i . Are closed immersions quasicompact? What about open immersions?

Exercise 3. Prove that a scheme X is the spectrum of a local ring if and only if it is quasicompact and has a unique closed point.

Exercise 4. Let $(X, \mathcal{O}_X)$ be a locally ringed space. Show that the following conditions are equivalent:

- (i) X is connected,
- (ii) 1 and 0 are the unique idempotents in $\mathcal{O}_X(X)$,
- (iii) $\text{Spec } \mathcal{O}_X(X)$ is connected.

Exercise 5. Let $\mathbb{F}$ be a field. Show that $\mathbb{A}_{\mathbb{F}}^1$ is infinite.

Exercise 6. Let $\mathbb{F}$ be a field. Let $\phi: A \rightarrow B$ be a homomorphism of finitely generated $\mathbb{F}$ -algebras. Show that the preimage along ϕ of a maximal ideal $\mathfrak{m} \subset B$ is a maximal ideal in A . That is, $f_{\phi}: \text{Spec } B \rightarrow \text{Spec } A$ maps closed points to closed points.

Exercise 7. Let $\phi: A \rightarrow B$ be a ring homomorphism. Let $f_{\phi}: \text{Spec } B \rightarrow \text{Spec } A$ be the induced morphism of affine schemes. Show that f_{ϕ} is *dominant* (i.e. its image is dense) if and only if $\ker \phi \subset \sqrt{0_A}$.

Exercise 8. Let $\mathbb{F}$ be a field. In $\mathbb{F}[x, y]$, compute the radical of $I = (x^3 - y^6, xy - y^3)$.

Exercise 9. Show that $V(y^2 - x^3) \subset \mathbb{A}_{\mathbf{k}}^2$ is irreducible.

Exercise 10. Describe and draw $\text{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$.