ALGEBRAIC GEOMETRY 2026/27 — EXERCISE SHEET #3

Exercise 1 (Towards tangent spaces)**.** Let $\mathbb{F}$ be a field, set $R = \mathbb{F}[t]/t^2$, and let

$$X = \operatorname{Spec}\mathbb{F}[x_1,\dots,x_n]/(f_1,\dots,f_m).$$

Fix a rational point $a = (a_1,\dots,a_n) \in X(\mathbb{F})$. Show that the $\mathbb{F}$-morphisms $\operatorname{Spec} R \to X$ whose restriction along $\operatorname{Spec}\mathbb{F} \hookrightarrow \operatorname{Spec} R$ is a are in bijection with the vectors $b = (b_1,\dots,b_n) \in \mathbb{F}^n$ satisfying

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(a) b_i = 0, \qquad 1 \le j \le m.$$

Exercise 2. Let X be obtained by gluing $\operatorname{Spec}\mathbb{F}[t]$ and $\operatorname{Spec}\mathbb{F}[u]$ along their punctured lines by $u \mapsto t$.

(i) Show that

$$\mathscr{O}_X(X) = \left\{ (a(t), b(u)) \;\middle|\; a(t) = b(t) \text{ in } \mathbb{F}[t,t^{-1}] \right\} \cong \mathbb{F}[t].$$

(iii) The coordinate maps on the two charts glue to a morphism $q\colon X \to \mathbb{A}^1_{\mathbb{F}}$. Show that q identifies the two origins and induces an isomorphism on global sections. Deduce that X is not affine.

Exercise 3. Prove that $\mathbb{P}^n(\mathbb{F})$ is naturally identified with the set of $\mathbb{F}$-rational points of the scheme $\mathbb{P}^n_{\mathbb{F}}$.

Exercise 4. Set $P = (0:0:1:0) \in \mathbb{P}^3_{\mathbb{C}}$. Let $Y \subset \mathbb{P}^3_{\mathbb{C}}$ be the twisted cubic. Compute the (set-theoretic) image of Y along the projection $\mathbb{P}^3_{\mathbb{C}} \setminus \{P\} \to \mathbb{P}^2_{\mathbb{C}}$.

Exercise 5. Let $X = \operatorname{Spec}\mathbf{k}[x,y,z]/I \subset \mathbb{A}^3_{\mathbf{k}}$, where $I = (x^2 - yz, y^2 - xz) \subset \mathbf{k}[x,y,z]$. Compute the irreducible components of X.

Exercise 6 (Integrality of hypersurfaces)**.** Let $f \in \mathbb{F}[x_1,\dots,x_n]$ be a nonconstant polynomial. Show that $\operatorname{Spec}\mathbb{F}[x_1,\dots,x_n]/(f)$ is reduced (resp. irreducible, resp. integral) if and only if f is square-free (resp. admits exactly one irreducible factor up to associates, resp. is irreducible).

Exercise 7 (BLA $\neq$ Geometrically BLA)**.** Let $\mathbb{F}$ be a field. Find examples of connected (resp. irreducible, resp. integral) $\mathbb{F}$-schemes X such that $X \times_{\mathbb{F}} \mathbb{E}$ is not connected (resp. irreducible, resp. integral) for some finite extension $\mathbb{E} \supset \mathbb{F}$.

Andrea T. Ricolfi, aricolfi@sissa.it