Computations in Enumerative Geometry

Andrea T. Ricolfi

These notes are for exclusive use of the SISSA PhD Students

Academic Year 2025–2026

Contents

0 | Introduction

The typical question in classical enumerative geometry asks how many objects satisfy a given list of geometric conditions. The presence of this 'list' makes the subject tightly linked to the subject of intersection theory.

Examples of classical problems in the subject are the following:

(1) Given an integer $n > 0$, how many lines $\ell \subset \mathbb{P}^{n+1}$ are incident to $2n$ general $(n-1)$-planes $\Lambda_1, \ldots, \Lambda_{2n} \subset \mathbb{P}^{n+1}$? (Answer in **??**)

(2) How many lines $\ell \subset \mathbb{P}^3$ lie on a general cubic surface $S \subset \mathbb{P}^3$? (Answer in **??**)

(3) How many lines $\ell \subset \mathbb{P}^4$ lie on a general quintic 3-fold $Y \subset \mathbb{P}^4$? (Answer in **??**)

In the questions above the objects we want to count are lines in some projective space, on which we impose some geometric constraints, such as intersecting other linear spaces or lying on a smooth hypersurface. In (1), we impose incidence with $2n$ linear spaces. Why exactly $2n$? We immediately see that in order to even get started we have to ask ourselves the following:

> **Question 1**. How do we know how many constraints we should put on our objects in order to *expect* a finite answer? In other words, how do we ask the right question?

See Section 1.1 for a full treatment of the topic 'expectations' in the case of lines on hypersurfaces: we will confirm that the expected number of lines on a general hypersurface $Y \subset \mathbb{P}^n$ of degree d is finite precisely when $d = 2n-3$.

The main idea to guide our geometric intuition in formulating and solving an enumerative problem should be the following recipe:

(A) construct a moduli[1] space $\mathcal{M}$ for the objects we are interested in,

(B) compactify $\mathcal{M}$ if necessary,

(C) impose $\dim \mathcal{M}$ conditions to expect a finite number of solutions, and

(D) count these solutions via intersection theory methods.

None of these steps is a trivial one, in general. The last two, in a little more detail, would ideally go as follows: each 'condition' we impose in step (C) is described by a cycle $Z_i \subset \mathcal{M}$ which is Poincaré dual to a Chow class $\alpha_i \in A^*\mathcal{M}$, and the intersection of these cycles is represented by the product

[1] The latin word *modulus* means *parameter*, and its plural is *moduli*. Thus a *moduli space* is to be thought of as a parameter space for objects of some kind.

$\alpha = \alpha_1 \cup \cdots \cup \alpha_r \in A^{\dim \mathcal{M}} \mathcal{M}$, where '$\cup$' is the ring multiplication in the Chow ring $A^* \mathcal{M}$.[2] The fact that α lies is the top degree (i.e. in maximal codimension) means that

$$\sum_{i=1}^{r} \operatorname{codim}(Z_i, \mathcal{M}) = \dim \mathcal{M},$$

which we should have achieved in the third step if we have 'asked the right question'. The final step asks us to compute the number

$$\int_{\mathcal{M}} \alpha = \deg_{\mathcal{M}}(\alpha \cap [\mathcal{M}]) \in \mathbb{Z},$$

where $\deg_{\mathcal{M}} \colon A_* \mathcal{M} \to A_* \mathrm{pt} = \mathbb{Z}$ is the pushforward on Chow groups along the structure morphism $\mathcal{M} \to \mathrm{pt}$, which exists by compactness of $\mathcal{M}$.

We are therefore facing the following question:

> **Question 2.** How do we know this intersection-theoretic degree is the answer to our original question? In other words, how to ensure that our algebraic solution is actually *enumerative*?

Concretely, take Problem (2): how do we make sure that each line $\ell \subset S$ appears as a point in the moduli space with multiplicity one? The truth is that we cannot *always* be sure that this is the case. It will be, both for Problem (2) and Problem (3) (by Lemma 1.1.4 and **??** respectively), but not in general. However, we should get used to the idea that this is not something to be worried about: if a solution comes with multiplicity bigger than one, there must be some good geometric reason for this, and we should not disregard it. More precisely (but not too precisely), if a point on the moduli space is 'fat', i.e. nonreduced, it means that the geometric object it corresponds to has nontrivial deformations, and thus it is natural to count it with some multiplicity — in this sense, one may say that our original enumerative question was too naive.

Remark 0.0.1. Compactness of $\mathcal{M}$ (in the above example, the Grassmannian) is used in order to make sense of taking the *degree* of cycles. Intuitively, we need compactness in order to prevent the solutions of our enumerative problem to escape to infinity, like for instance it would occur if we were to intersect two *parallel* lines in $\mathbb{A}^2$.

We emphasise that compatcness really is a non-negotiable condition we have to ask of our moduli space — with an important exception, that will be treated in later sections: the case when the moduli space has a torus action. In this case, if the torus fixed locus $\mathcal{M}^{\mathbb{T}} \subset \mathcal{M}$ is compact, a sensible enumerative solution to a counting problem can be *defined* by means of the *localisation formula*, one of the most important tools in enumerative geometry (and in these notes). The original formula due to Atiyah and Bott will be proved in **??**.

One more fundamental notion in counting problems, *transversality*, is discussed in the next subsection, by means of an elementary example.

0.1 Transversality, and counting lines through two points

Let X be an algebraic variety, $x \in X$ a point. Denote by $T_x X = (\mathfrak{m}_x / \mathfrak{m}_x^2)^*$ the Zariski tangent space of X at x, where $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ is the maximal ideal in the local ring attached to x.

[2] The reader can more or less safely replace A_* with homology H_* throughout these notes.

Definition 0.1.1 (Transverse intersection)**.** Let Y be a quasiprojective variety. We say that subvarieties $Z_1,\ldots,Z_m$ of Y *intersect transversely* at a smooth point $y\in Y$ if y is a smooth point on each Z_i and $\operatorname{codim}(\bigcap_i T_yZ_i, T_yY)=\sum_i \operatorname{codim}(T_yZ_i,T_yY)$.

Consider the enumerative problem of counting the number of lines in $\mathbb{P}^2$ through two given points $p,q\in\mathbb{P}^2$. Let N_{pq} be this number. Then, of course,

$$\mathsf{N}_{pq}=\begin{cases}1 & \text{if } p\neq q,\\ \infty & \text{if } p=q.\end{cases}$$

The answer $\mathsf{N}_{pp}=\infty$ corresponds to the cardinality of the pencil $Z_p\cong\mathbb{P}^1$ of lines through p (see Figure 1). For the sake of 'completeness', the formula $\mathsf{N}_{pq}=1$ will be proved in **??**.

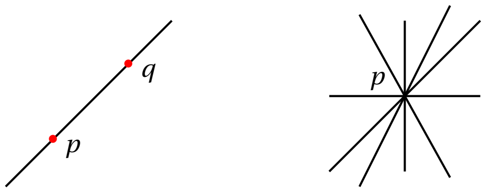


Figure 1: The unique line through two distinct points, and the infinitely many lines through one point in the plane.

The point we want to make in this section is that

$$\boxed{\text{the answer '1' can be recovered in the degenerate setting } p=q.}$$

Remark 0.1.2 (Let us not cheat)**.** To obtain the answer relative to the picture on the left, we have to notice that the two cycles

$$Z_p=\left\{\,\ell\subset\mathbb{P}^2\,\middle|\,p\in\ell\,\right\}\subset\mathbb{P}^{2*},\quad Z_q=\left\{\,\ell\subset\mathbb{P}^2\,\middle|\,q\in\ell\,\right\}\subset\mathbb{P}^{2*}$$

intersect transversely in the dual projective space $\mathbb{P}^{2*}$, and we can use the intersection product on $\mathbb{P}^{2*}$ to compute $Z_p\cdot Z_q=\#(Z_p\cap Z_q)=1$. Now, using basic intersection theory, it is clear how to obtain the answer '1' also in the case $p=q$. Since we are working in $\mathbb{P}^{2*}\cong\mathbb{P}^2$, we know that any $q\neq p$ yields a *homologous* cycle $Z_q\sim Z_p$, and again the intersection product yields $Z_p^2=Z_p\cdot Z_q=1$. But in general we will not be working in such a pleasant ambient space and thus we will not know whether algebraic deformations such as $Z_p\rightsquigarrow Z_q$, leading to a transverse setup, are available. For instance, a (-1)-curve[3] C on a surface S cannot be 'moved' algebraically on S to another curve C' such that C and C' intersect transversely!

Now, the case $p=q$ is a 'degeneration' of the case $p\neq q$, and we certainly want our enumerative answer not to depend on small perturbations of the geometry of the problem. Why do we want that? Just because we are reasonable people: we were already taught how to be reasonable when our first Calculus teacher told us that a decent function is *at least* continuous.[4]

Next we explain how to get the 'correct' answer

$$\mathsf{N}_{pp}^{\text{corrected}}=1$$

[3]A (-1)-curve on a surface S is a curve $C\subset S$ such that $C.C=-1$, where the intersection number $C.C$ can be seen as the degree of the normal bundle $\mathscr{N}_{C/S}=\mathscr{O}_S(C)|_C$ to C in S.

[4]Ultimately, we are going to study *invariants*, e.g. Donaldson–Thomas invariants. They deserve to be called that precisely because they don't change if we slightly (but holomorphically) deform the variety they are attached to.

by means of the *excess intersection formula*, one of the most important tools in classical enumerative geometry. We mention it not only because it is a beautiful piece of intersection theory, but also because it lies at the very roots of *modern* enumerative geometry, lying right at the foundation of the idea of virtual classes.

Before we start, we need to recall the following important notion.

Definition 0.1.3 ((Co)normal sheaf)**.** The *conormal sheaf* of a closed immersion of schemes $X \hookrightarrow M$ defined by an ideal $\mathscr{I} \subset \mathscr{O}_M$ is the quasicoherent[5] $\mathscr{O}_X$-module

$$\mathscr{C}_{X/M} = \mathscr{I}/\mathscr{I}^2,$$

and the *normal sheaf* is its $\mathscr{O}_X$-linear dual,

$$\mathscr{N}_{X/M} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_X).$$

The sheaves $\mathscr{C}_{X/M}$ and $\mathscr{N}_{X/M}$ are locally free (of rank d) when $X \hookrightarrow M$ is a regular immersion (of codimension d).

Example 0.1.4. If $X \hookrightarrow M = \mathbb{P}^r$ is a hypersurface of degree d, then the ideal sheaf of X in $\mathbb{P}^r$ is the invertible sheaf $\mathscr{O}_{\mathbb{P}^r}(-d)$, so $\mathscr{N}_{X/\mathbb{P}^r} = \mathscr{O}_{\mathbb{P}^r}(d)|_X$.

Notation 0.1.5. Let $X \hookrightarrow M$ be a closed immersion. We set

$$N_{X/M} = \operatorname{Spec}_{\mathscr{O}_X} \operatorname{Sym} \mathscr{C}_{X/M}.$$

It is naturally a scheme over X. With a slight terminology abuse, we will also refer to it as the *normal sheaf* to X in M.

Exercise 0.1.6. Let $X \hookrightarrow M$ be a closed immersion, $\widetilde{M} \to M$ a morphism, set $\widetilde{X} = X \times_M \widetilde{M}$ and let $g\colon \widetilde{X} \to X$ be the induced map. Show that there is a natural injective map of sheaves $\mathscr{N}_{\widetilde{X}/\widetilde{M}} \hookrightarrow g^*\mathscr{N}_{X/M}$, which is an isomorphism whenever $\widetilde{M} \to M$ is flat. Deduce that there is a closed immersion

$$N_{\widetilde{X}/\widetilde{M}} \hookrightarrow g^* N_{X/M} = N_{X/M} \times_X \widetilde{X}$$

of schemes over $\widetilde{X}$. (**Hint**: Try to construct a surjection $g^*\mathscr{C}_{X/M} \twoheadrightarrow \mathscr{C}_{\widetilde{X}/\widetilde{M}}$ involving the conormal sheaves. If in need of further hints, see [41, Tag 01R1]).

Also recall (see [27, II.8.13] for a reference) that on any projective space $\mathbb{P}^r$ we have the *Euler sequence*

$$0 \to \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to \mathscr{T}_{\mathbb{P}^r} \to 0, \tag{0.1.1}$$

where $\mathscr{O}_{\mathbb{P}^r}(1)$ is the hyperplane bundle and $\mathscr{T}_{\mathbb{P}^r}$ is the algebraic tangent bundle.

Now back to our problem. The $\mathbb{P}^1$ of lines through p can be neatly seen as the exceptional divisor E in the blowup $B = \operatorname{Bl}_p \mathbb{P}^2$, cf. Figure 2.

Looking at the fibre diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\pi"] \\ p \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd} \tag{0.1.2}$$

[5] It is coherent as long as M is locally noetherian.

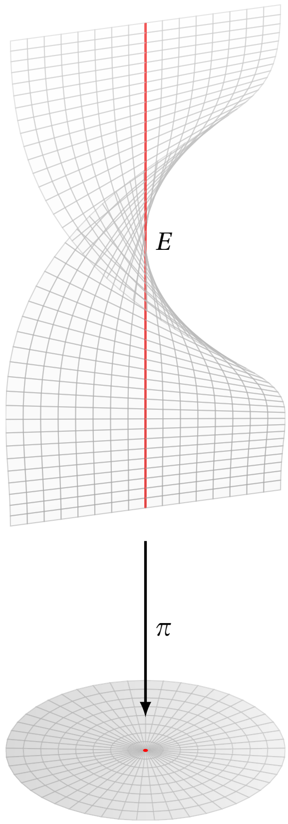


Figure 2: The blowup $\pi\colon B \to \mathbb{P}^2$ of the plane in one point.

we know by Exercise 0.1.6 that there is an injection of locally free sheaves $\mathcal{N}_{E/B} = \mathcal{O}_E(-1) \subset g^*\mathcal{N}_{p/\mathbb{P}^2}$. The *excess bundle* (or *obstruction bundle*)

$$\mathrm{Ob}_{pp} \to \mathbb{P}^1$$

of the fibre diagram (0.1.2) is defined as the quotient of these two bundles [22, Section 6.3]. But the short exact sequence

$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E \otimes T_p\mathbb{P}^2 \to \mathrm{Ob}_{pp} \to 0$$

is nothing but the Euler sequence (0.1.1) on $\mathbb{P}^1$ twisted by $\mathcal{O}_E(-1)$. Therefore

$$\mathrm{Ob}_{pp} = \mathcal{T}_{\mathbb{P}^1}(-1) = \mathcal{O}_{\mathbb{P}^1}(2-1) = \mathcal{O}_{\mathbb{P}^1}(1).$$

Note that we can repeat the process with $q \neq p$, which would yield $\pi^{-1}(q) = \mathrm{pt}$. In this case we get $\mathrm{Ob}_{pq} = 0$. We can now write a universal formula for our counting problem: if $\mathcal{M}_{pq} = \pi^{-1}(q)$ is the 'moduli space of lines' through p and q, the *virtual number* of lines through p and q is

$$\int_{\mathcal{M}_{pq}} e(\mathrm{Ob}_{pq}) = 1,$$

for all $(p,q) \in \mathbb{P}^2 \times \mathbb{P}^2$. The *Euler class* $e(V)$ of a vector bundle V is its top Chern class. Note that the rank of Ob_{pq} is the difference between the actual dimension of the moduli space, and the expected one.

Remark 0.1.7. Note that, unlike in Remark 0.1.2, we have now obtained

$$\mathsf{N}_{pp}^{\text{corrected}} = 1$$

as an intersection number on the *actual* moduli space $Z_p \cong \mathbb{P}^1 \cong E$.

This discussion allows us to formulate another intrinsic difficulty in enumerative geometry. Suppose, just to dream for a second, that we are able to solve *all* enumerative problems in generic (transverse) situations, and we know that the answer does not change after a small perturbation of the initial data.

> **Question 3**. How do we 'pretend' we can work in a transverse situation when there is none available?

The modern way to do this is to use *virtual fundamental classes*.

1 | Moduli spaces

1.1 Warming up: lines on hypersurfaces

Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree d. We want to show the following:

$$\boxed{\begin{array}{c}\text{We should expect a finite number of lines on } Y \text{ if and only if } d = 2n-3.\\ \text{We should expect \textit{no lines} on } Y \text{ if } d > 2n-3.\\ \text{We should expect infinitely many lines on } Y \text{ if } d < 2n-3.\end{array}}$$

To understand the condition

$$\ell \subset Y$$

for a given line $\ell \subset \mathbb{P}^n$ and a hypersurface $Y \subset \mathbb{P}^n$, we give the following concrete example.

Example 1.1.1. Let $\ell \subset \mathbb{P}^3$ be the line cut out by $L_1 = L_2 = 0$, where $L_i = L_i(z_0, z_1, z_2, z_3)$ are linear forms on $\mathbb{P}^3$. To fix ideas, set $L_1 = z_0$ and $L_2 = z_0 + z_2 + z_3$. Let $Y \subset \mathbb{P}^3$ be defined by a homogeneous equation $f = 0$, for instance the cubic polynomial

$$f = z_0^3 + 3z_0 z_1^2 - z_2^2 z_3.$$

Then we see that plugging $L_1 = L_2 = 0$ into f does not give zero, for

$$f\big|_\ell = 0 + 0 - z_2^2(-z_0 - z_2) = z_2^3.$$

This means that ℓ is not contained in Y. On the other hand, the line cut out by L_1 and $L_2' = z_3$ lies entirely on Y.

Let $Y \subset \mathbb{P}^n$ be the zero locus of a general homogeneous polynomial

$$f \in \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)).$$

As we anticipated in Example 1.1.1, a line $\ell \subset \mathbb{P}^n$ is contained in Y if and only if $f|_\ell = 0$. This condition can be rephrased by saying that the image of f under the restriction map

$$\mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \xrightarrow{\mathsf{res}_\ell} \mathrm{H}^0(\ell, \mathscr{O}_\ell(d)) \tag{1.1.1}$$

vanishes. We want to determine when we should expect Y to contain a finite number of lines. We set, informally,

$$\mathsf{N}_1(Y) = \text{expected number of lines in } Y.$$

Let us consider the Grassmannian

$$\mathbb{G} = \mathsf{Gr}(2, n+1) = \mathbb{G}(1, n) = \left\{ \text{Lines } \ell \subset \mathbb{P}^n \right\},$$

a smooth complex projective variety of dimension $2n-2$. Recall the universal structures living on $\mathbb{G}$. First of all, the tautological exact sequence

$$\begin{array}{ccccccccc} & & \text{rank } 2 & & \text{rank } n+1 & & \text{rank } n-1 & & \\ & & \Big| & & \Big| & & \Big| & & \\ 0 & \longrightarrow & \mathscr{S} & \longrightarrow & \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^* & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \end{array}$$

where the fibre of $\mathscr{S}$ over a point $[\ell] \in \mathbb{G}$ is the 2-dimensional vector space $\mathrm{H}^0(\ell, \mathscr{O}_\ell(1))^*$. Let, also,

$$\mathcal{L} = \left\{ (p,[\ell]) \in \mathbb{P}^n \times \mathbb{G} \,\middle|\, p \in \ell \right\} \subset \mathbb{P}^n \times \mathbb{G}$$

be the universal line. Consider the two projections

$$\begin{tikzcd} \mathcal{L} \arrow[r, "q"] \arrow[d, "\pi"'] & \mathbb{P}^n \\ \mathbb{G} & \end{tikzcd}$$

and the coherent sheaf

$$\mathscr{E}_d = \pi_* q^* \mathscr{O}_{\mathbb{P}^n}(d).$$

Exercise 1.1.2. Show that $\mathscr{E}_d$ is locally free of rank $d+1$, and that one has an isomorphism of locally free sheaves

$$\mathscr{E}_d \cong \mathsf{Sym}^d \mathscr{S}^*.$$

(**Hint**: use cohomology and base change, e.g. [16, Theorem B.5]).

Dualising the universal inclusion $\mathscr{S} \hookrightarrow \mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))^*$ and applying Sym^d, we obtain a surjection

$$\mathscr{O}_{\mathbb{G}} \otimes_{\mathbb{C}} \mathrm{H}^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \twoheadrightarrow \mathsf{Sym}^d \mathscr{S}^*,$$

which is just a global version of (1.1.1). The association

$$\mathbb{G} \ni [\ell] \mapsto f|_\ell \in \mathrm{H}^0(\ell, \mathscr{O}_\ell(d)) \cong \mathsf{Sym}^d \mathrm{H}^0(\ell, \mathscr{O}_\ell(1))$$

defines a section τ_f of $\mathscr{E}_d \to \mathbb{G}$. The zero locus of $\tau_f = \pi_* q^* f \in \mathrm{H}^0(\mathbb{G}, \mathscr{E}_d)$ is the locus of lines contained in Y.

Definition 1.1.3 (Fano scheme of lines)**.** Let $Y \subset \mathbb{P}^n$ be a hypersurface defined by $f = 0$. The zero scheme

$$F_1(Y) = Z(\tau_f) \hookrightarrow \mathbb{G} = \mathbb{G}(1,n)$$

is called the *Fano scheme of lines* in Y.

Since f is generic, $\tau_f \in \Gamma(\mathbb{G}, \mathscr{E}_d)$ is also generic. Therefore, the fundamental class of the Fano scheme of lines in Y is Poincaré dual to the Euler class

$$e(\mathscr{E}_d) \in A^{d+1}\mathbb{G}.$$

Thus $[F_1(Y)] = e(\mathscr{E}_d) \cap [\mathbb{G}] \in A_*\mathbb{G}$ is a 0-cycle if and only if $d+1 = 2n-2$, i.e.

$$d = 2n-3.$$

A rigorous definition of $\mathsf{N}_1(Y)$ is then

$$\mathsf{N}_1(Y) = \int_{\mathbb{G}} e(\mathscr{E}_d) = \int_{\mathbb{G}} c_{d+1}(\mathsf{Sym}^d\,\mathscr{S}^*).$$

Such degree is the *actual* number of lines on Y whenever $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/Y}) = 0$ for all $\ell \subset Y$. This condition means that the Fano scheme is reduced at all its points $[\ell]$, since $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/Y})$ is its tangent space at the point $[\ell]$.

LEMMA 1.1.4. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $\ell \subset S$ is a line, then* $\mathrm{H}^0(\ell, \mathscr{N}_{\ell/S}) = 0$.

Proof. It is enough to show that $\mathscr{N} = \mathscr{N}_{\ell/S}$, viewed as a line bundle on $\ell \cong \mathbb{P}^1$, has negative degree. By the adjunction formula,

$$K_\ell = K_S|_\ell \otimes_{\mathscr{O}_\ell} \mathscr{N}.$$

Using that $K_\ell = \mathscr{O}_\ell(-2)$ and $K_S = K_{\mathbb{P}^3}|_S \otimes_{\mathscr{O}_S} \mathscr{N}_{S/\mathbb{P}^3} = \mathscr{O}_S(d-4)$ for a surface of degree d in $\mathbb{P}^3$, by taking degrees we obtain

$$-2 = (3-4) + \deg \mathscr{N},$$

so that $\deg \mathscr{N} = -1 < 0$. $\square$

Exercise 1.1.5. Let $Y \subset \mathbb{P}^n$ be a general hypersurface of degree $d \leqslant 2n-3$. Show that $F_1(Y) \subset \mathbb{G}(1,n)$ is smooth of dimension $2n-3-d$.

1.2 Hilbert and Quot schemes of points

1.2.1 Existence, and Quot-to-Chow

Let X be a smooth quasiprojective variety of dimension $d \geqslant 1$, and let F be a locally free sheaf of rank $r \geqslant 1$ over X. Fix an integer $n \geqslant 0$. The *Quot functor* attached to this data is the functor $\underline{\mathsf{Quot}}_X(F,n)\colon \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Sets}$ sending a $\mathbb{C}$-scheme B to the set of isomorphism classes of surjections $F_B \twoheadrightarrow T$ in $\mathsf{Coh}(X \times B)$, where T is a B-flat sheaf such that T_b is 0-dimensional and satisfies $\chi(T_b) = n$ for every geometric point $b \in B$. Two surjections are considered isomorphic if they share the same kernel. One can also say that the functor parametrises *short exact sequences*

$$0 \longrightarrow K \longrightarrow F_B \longrightarrow T \longrightarrow 0$$

in $\mathsf{Coh}(X \times B)$, with no equivalence relation imposed. A classical result of Grothendieck states that this functor is represented by a quasiprojective scheme (projective as soon as X is projective)

$$\mathsf{Quot}_X(F,n).$$

If $F = \mathscr{O}_X$, we obtain the *Hilbert scheme of points*

$$\mathsf{Hilb}^n(X) = \mathsf{Quot}_X(\mathscr{O}_X, n).$$

It parametrises closed subschemes $Z \subset X$ such that $\chi(\mathscr{O}_Z) = n$. Its B-valued points, namely the short exact sequences

$$0 \longrightarrow \mathscr{I}_{\mathcal{Z}} \longrightarrow \mathscr{O}_{X\times B} \longrightarrow \mathscr{O}_{\mathcal{Z}} \longrightarrow 0,$$

with $\mathscr{O}_{\mathcal{Z}}$ flat over B of relative degree n, correspond precisely to closed subschemes $\mathcal{Z} \hookrightarrow X \times B$ such that $\mathcal{Z} \to B$ is flat of degree n.

Notation 1.2.1. The closed point in $\mathsf{Hilb}^n(X)$ corresponding to a closed subscheme $Z \subset X$ with ideal sheaf $\mathscr{I}_Z \subset \mathscr{O}_Z$ will be denoted by $[Z]$ or by $[\mathscr{I}_Z]$.

Morphisms *into* the Quot scheme are classified by the Quot functor, thanks to its representability. The most important morphism *out* of the Quot scheme is the *Quot-to-Chow morphism*, namely the morphism

$$\mathsf{Quot}_X(F,n) \xrightarrow{\sigma_{F,n}} \mathsf{Sym}^n(X) = X^n/\mathfrak{S}_n, \tag{1.2.1}$$

taking the class of a quotient $[F \twoheadrightarrow T]$ to the 0-cycle

$$[\operatorname{Supp} T] = \sum_{x\in X} \mathsf{length}_{\mathscr{O}_{X,x}} T_x \cdot x.$$

See [40, Cor. 7.15] or [20, 19] for the construction. The *punctual Quot scheme*

$$\mathsf{Quot}_X(F,n)_x \subset \mathsf{Quot}_X(F,n)$$

is defined to be the fibre of $\sigma_{F,n}$ over the 'fat cycle' $[nx] \in \mathsf{Sym}^n(X)$. If F has rank 1, we denote it by $\mathsf{Hilb}^n(X)_x$.

Definition 1.2.2. The *smoothable component* $\Gamma_{\mathsf{sm}} \subset \mathsf{Hilb}^n(X)$ is the closure of the locus of reduced subschemes $Z \subset X$. If $X = \mathbb{A}^d$, we denote it by $\Gamma^{n,d}_{\mathsf{sm}}$.

PROPOSITION 1.2.3 ([8, Prop. 4.15]). *All monomial ideals $I \subset \mathbb{C}[x_1,\ldots,x_d]$ of colength n lie in the smoothable component $\Gamma^{n,d}_{\mathsf{sm}} \subset \mathsf{Hilb}^n(\mathbb{A}^d)$.*

1.2.2 What's (un)known?

Not much is known about the geometry of $\mathsf{Hilb}^n(X)$ for $d \geqslant 3$. Here is a recap on some known properties of Hilbert and Quot scheme of points:

1. $\mathsf{Hilb}^n(\mathbb{A}^d)$ is connected for all n and d. It is smooth (and irreducible) if and only if $d \leqslant 2$ or $n \leqslant 3$.
2. $\mathsf{Hilb}^n(\mathbb{A}^d)$ is irreducible for all d and $n \leqslant 7$, see [35]. If $d \geqslant 4$, then $\mathsf{Hilb}^n(\mathbb{A}^d)$ is irreducible if and only if $n \leqslant 7$.
3. $\mathsf{Hilb}^n(\mathbb{A}^3)$ is irreducible for $n \leqslant 11$, see [28, 15] and the references therein. See also [45] for $n = 9, 10$. Moreover, $\mathsf{Hilb}^n(\mathbb{A}^3)$ is reducible for $n \geqslant 78$, see [29].
4. $\mathsf{Hilb}^{13}(\mathbb{A}^6)$ is nonreduced by Szachniewicz's work [43].
5. The punctual Hilbert scheme $\mathsf{Hilb}^n(\mathbb{A}^2)_0$ is irreducible of dimension $n-1$, see [7, Corollaire V.3.3].
6. The punctual Hilbert scheme $\mathsf{Hilb}^n(\mathbb{A}^3)_0$ is irreducible for all $n \leqslant 11$, see [30].
7. If F is locally free, then $\mathsf{Quot}_X(F,n)$ is connected [37, Theorem 1.4].
8. If $r > 1$, then $\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r},n)$ is smooth if and only if $n = 1$.
9. $\mathsf{Quot}_{\mathbb{A}^2}(\mathscr{O}^{\oplus r},n)$ is irreducible of dimension $(r+1)n$, see [17].
10. If $d, r > 3$, then $\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r},8)$ has a generically nonreduced component [31, §6.5].

On the other hand, the following questions are open since a long time:

1. What is the smallest n such that $\mathsf{Hilb}^n(\mathbb{A}^3)$ is reducible?
2. If $d \geqslant 3$, what is an upper bound for the number of irreducible components of $\mathsf{Hilb}^n(\mathbb{A}^d)$?
3. If $d \geqslant 3$ and $p \in \Gamma^{n,d}_{\mathsf{sm}} \subset \mathsf{Hilb}^n(\mathbb{A}^d)$ belongs to the smoothable component, what is, at most, the dimension of the tangent space of $\mathsf{Hilb}^n(\mathbb{A}^d)$ at p?
4. Is $\mathsf{Hilb}^n(\mathbb{A}^3)$ generically reduced? That is, if $W \subset \mathrm{H} = \mathsf{Hilb}^n(\mathbb{A}^3)$ is an irreducible component with generic point ξ, is $\mathscr{O}_{\mathrm{H},\xi}$ reduced?

1.2.3 The local case

Let us set $X = \mathbb{A}^d$, so that necessarily F is a trivial bundle (of rank r). We now give a hands-on (scheme-theoretic) construction of

$$\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) = \left\{ K \subset \mathbb{C}[x_1,\ldots,x_d]^{\oplus r} \,\middle|\, \dim_{\mathbb{C}} \mathbb{C}[x_1,\ldots,x_d]^{\oplus r}/K = n \right\},$$

that will turn out useful later. More precisely, we will provide equations cutting out the Quot scheme inside a smooth quasiprojective variety (the so-called *noncommutative Quot scheme*), cf. Theorem 1.2.7.

Let us get started with our description of $\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$. To ease notation, let us put $R_d = \mathbb{C}[x_1,\ldots,x_d]$. The condition defining a point $[K] \in \mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$ is that the R_d-linear quotient

$$R_d^{\oplus r} \twoheadrightarrow R_d^{\oplus r}/K$$

is a vector space of dimension n. To construct a point in the Quot scheme, we need:

(1) a vector space $V_n \cong \mathbb{C}^n$,

(2) an R_d-module structure

$$\vartheta\colon R_d \longrightarrow \mathrm{End}_{\mathbb{C}}(V_n)$$

with the property that

(3) such structure is induced by an R_d-linear surjection from $R_d^{\oplus r}$.

So let us fix an n-dimensional vector space V_n. Later we will have to remember that we made such a choice, and since all we wanted was "$\dim V_n = n$" we will have to quotient out all equivalent choices. Let us forget about this for the moment. In (2), we need ϑ to be a ring homomorphism, so we need to specify one endomorphism of V_n for each coordinate $x_i \in R_d$. All in all, ϑ gives us d matrices

$$A_1, A_2, \ldots, A_d \in \mathrm{End}_{\mathbb{C}}(V_n).$$

The matrix A_i will be responsible for the R_d-linear operator "multiplication by x_i" for the resulting module structure on V_n. Also in this step we should note a reminder for later: strictly speaking, what we have defined so far is a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module structure on V_n. But in R_d the variables commute with one another. So we will eventually have to impose the relations $[A_i, A_j] = 0$ for all $1 \leqslant i < j \leqslant d$.

Condition (3) is a little more tricky. Let us reason backwards, assuming we already have an R_d-linear quotient $\phi\colon R_d^{\oplus r} \twoheadrightarrow V_n$. Then it is clear that the images of the coordinate vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r \in R_d^{\oplus r}$ must generate V_n as a $\mathbb{C}\langle x_1, x_2, \ldots, x_d\rangle$-module. In other words, every element $w \in V_n$ belongs to

$$\mathsf{Span}_{\mathbb{C}} \left\{ A_1^{m_1} A_2^{m_2} \cdots A_d^{m_d} \cdot \phi(\boldsymbol{e}_i) \,\middle|\, m_i \geqslant 0,\ 1 \leqslant i \leqslant r \right\}.$$

This tells us exactly what we should add to the picture to obtain Condition (3): for a fixed $\mathbb{C}\langle x_1,x_2,\ldots,x_d\rangle$-module structure, i.e. d-tuple of matrices $(A_1,A_2,\ldots,A_d)$, we need to specify r vectors $v_1,\ldots,v_r\in V_n$ with the property that

$$\mathsf{Span}_{\mathbb{C}}\left\{ A_1^{m_1}A_2^{m_2}\cdots A_d^{m_d}\cdot v_i \,\middle|\, m_i\in\mathbb{Z}_{\geqslant 0},\, 1\leqslant i\leqslant r\right\} = V_n. \tag{1.2.2}$$

Let us consider the (dn^2+rn)-dimensional affine space

$$W_{r,d}^n = \mathrm{End}_{\mathbb{C}}(V_n)^{\oplus d}\oplus V_n^{\oplus r}. \tag{1.2.3}$$

Exercise 1.2.4. Show that the locus

$$U_{r,d}^n = \{(A_1,A_2,\ldots,A_d,v_1,\ldots,v_r)\,|\,(1.2.2)\text{ holds}\}\subset W_{r,d}^n$$

is a Zariski open subset.

Consider the GL_n-action on $W_{r,d}^n$ given by

$$g\cdot(A_1,A_2,\ldots,A_d,v_1,\ldots,v_r) = (A_1^g,A_2^g,\ldots,A_d^g,gv_1,\ldots,gv_r) \tag{1.2.4}$$

where $M^g = gMg^{-1}$ is the standard conjugation action.

LEMMA 1.2.5. *The* GL_n*-action* (1.2.4) *is free on* $U_{r,d}^n$*.*

Proof. If $g\in\mathrm{GL}_n$ fixes a point $(A_1,A_2,\ldots,A_d,v_1,\ldots,v_r)\in U_{r,d}^n$, then $v_i = gv_i$ lies in the invariant subspace $\ker(g-\mathrm{id})\subset V_n$ for $i=1,\ldots,r$. But by definition of $U_{r,d}^n$, the smallest invariant subspace containing $v_1,\ldots,v_r$ is V_n itself, thus $g=\mathrm{id}$. □

Definition 1.2.6 (Noncommutative Quot scheme)**.** The (geometric) GIT quotient

$$\mathsf{ncQuot}_{r,d}^n = U_{r,d}^n/\mathrm{GL}_n$$

is called the *noncommutative Quot scheme.*

Note that $\mathsf{ncQuot}_{r,d}^n$ is smooth by [41, Tag 02K5] and has dimension $\dim U_{r,d}^n - \dim\mathrm{GL}_n = (d-1)n^2+rn$.

The following result is of crucial importance for us (and can be taken as the *definition* of the local Quot scheme).

THEOREM 1.2.7 ([44, 1]). *There is a closed immersion*

$$\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r},n)\hookrightarrow \mathsf{ncQuot}_{r,d}^n$$

cut out by the relations $[A_i,A_j]=0$ *for* $1\leqslant i<j\leqslant d$*.*

Furthermore, if $d=3$*, such relations cut out the zero scheme of the exact 1-form*

$$\mathrm{d}f_{r,n}\in \mathrm{H}^0(\mathsf{ncQuot}_{r,3}^n,\Omega^1)$$

defined by the regular function $f_{r,n}\colon \mathsf{ncQuot}_{r,3}^n\to\mathbb{A}^1$ *sending*

$$[A_1,A_2,A_3,v_1,\ldots,v_r]\longmapsto \mathrm{Tr}\,A_1[A_2,A_3].$$

In particular, $\mathsf{Quot}_{\mathbb{A}^3}(\mathscr{O}^{\oplus r},n)$ *is a critical locus.*

Exercise 1.2.8. Let $d=r=1$. Show that $\mathsf{Hilb}^n(\mathbb{A}^1)=\mathsf{ncQuot}_{1,1}^n=\mathbb{A}^n$.

Remark 1.2.9. If $d=2$ the description of $\mathsf{Hilb}^n(\mathbb{A}^2)$ is equivalent to Nakajima's description [38, Theorem 1.14]. See also [28] for an equivalent description of $\mathsf{Hilb}^n(\mathbb{A}^d)$, in terms of *perfect extended monads.* See [9] for a description of $\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r},n)$ in terms of framed sheaves on $\mathbb{P}^d$, available when $d\geqslant 3$.

1.2.4 Quot-to-Chow in the local case

Fix $n \geqslant 0$ and $d, r \geqslant 1$. The Quot-to-Chow morphism

$$\sigma_{\mathscr{O}^{\oplus r},n} \colon \mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n) \longrightarrow \mathsf{Sym}^n \mathbb{A}^d$$

introduced in (1.2.1) can be reinterpreted as follows. Pick a point

$$[A_1, \ldots, A_d, v_1, \ldots, v_r] \in \mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)$$

and notice that since the matrices pairwise commute, they can be simultaneously made upper triangular. So, since the tuple is defined up to GL_n, we may assume they are in the form

$$A_\ell = \begin{pmatrix} a_{11}^{(\ell)} & * & * & \cdots & * \\ 0 & a_{22}^{(\ell)} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(\ell)} \end{pmatrix}, \quad 1 \leqslant \ell \leqslant d.$$

Then $\sigma_{\mathscr{O}^{\oplus r},n}$ is given by

$$[A_1, \ldots, A_d, v_1, \ldots, v_r] \longmapsto \sum_{1 \leqslant i \leqslant n} \left(a_{ii}^{(1)}, \ldots, a_{ii}^{(d)}\right).$$

When all the matrices are *nilpotent*, and only in this case, the corresponding quotient $\mathscr{O}^{\oplus r} \twoheadrightarrow T$ is entirely supported at the origin. In other words,

$$\mathsf{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_0 = \{ [A_1, \ldots, A_d, v_1, \ldots, v_r] \mid A_1, \ldots, A_d \text{ are nilpotent} \}$$

is a way to describe the punctual Quot scheme.

1.2.5 Smoothness

We shall make use of the following classical result from deformation theory.

PROPOSITION 1.2.10. *Let $p \in \mathsf{Quot}_X(F, n)$ be a closed point corresponding to a short exact sequence*

$$0 \longrightarrow K \longrightarrow F \longrightarrow T \longrightarrow 0.$$

Then

(i) *The tangent space of $\mathsf{Quot}_X(F, n)$ to p is the vector space*

$$T_p \, \mathsf{Quot}_X(F, n) = \mathrm{Hom}_X(K, T).$$

(ii) *If $\mathrm{Ext}^1_X(K, T) = 0$, then $\mathsf{Quot}_X(F, n)$ is smooth at p.*

PROPOSITION 1.2.11. *If X is a smooth quasiprojective surface, then $\mathsf{Hilb}^n(X)$ is smooth and irreducible of dimension $2n$. In particular, it is equal to the smoothable component.*

Proof. We may assume X to be projective, since the general result will follow from this one via a smooth compactification $X \hookrightarrow \overline{X}$, inducing an open immersion $\mathsf{Hilb}^n(X) \hookrightarrow \mathsf{Hilb}^n(\overline{X})$.

Suppose $\dim T_p \, \mathsf{Hilb}^n(X) = 2n$ for any $p \in \mathsf{Hilb}^n(X)$, then $\mathsf{Hilb}^n(X)$ is smooth along the smoothable component, because its interior is a smooth open subset of dimension $2n$. Since $\mathsf{Hilb}^n(X)$ is connected by [26], the presence of any other irreducible component W would yield singularities along $W \cap \Gamma_{\mathsf{sm}}$. Thus we are reduced to showing that $\dim T_p \operatorname{Hilb}^n(X) = 2n$ for any $p \in \operatorname{Hilb}^n(X)$.

Let $Z \subset X$ be the closed subscheme corresponding to p. By Proposition 1.2.10, we have $T_p \operatorname{Hilb}^n(X) = \operatorname{Hom}_X(\mathscr{I}_Z, \mathscr{O}_Z)$, so we get an exact sequence

$$0 \to \operatorname{Hom}_X(\mathscr{O}_Z, \mathscr{O}_Z) \xrightarrow{e} \operatorname{Hom}_X(\mathscr{O}_X, \mathscr{O}_Z) \xrightarrow{u} T_p \operatorname{Hilb}^n(X) \xrightarrow{v} \operatorname{Ext}^1_X(\mathscr{O}_Z, \mathscr{O}_Z)$$

where e is an isomorphism between n-dimensional vector spaces, so that $u = 0$ and v is injective. We next show that $\operatorname{ext}^1_X(\mathscr{O}_Z, \mathscr{O}_Z) \leqslant 2n$. Note that $\chi(\mathscr{O}_Z, \mathscr{O}_Z) = \operatorname{ext}^0(\mathscr{O}_Z, \mathscr{O}_Z) - \operatorname{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) + \operatorname{ext}^2(\mathscr{O}_Z, \mathscr{O}_Z) = 2n - \operatorname{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z)$, where we used Serre duality to compute ext^2. Note that, for any locally free sheaf E on X, one has $\chi(E, \mathscr{O}_Z) = \dim \mathrm{H}^0(Z, \mathscr{O}_Z)^{\operatorname{rk} E} = n \cdot \operatorname{rk} E$. So, let $E^\bullet \to \mathscr{O}_Z$ be a resolution consisting of finitely many locally free sheaves E^i on X, so that $0 = \operatorname{rk} \mathscr{O}_Z = \operatorname{rk} E^\bullet = \sum_\ell (-1)^\ell \operatorname{rk} E^\ell$. This implies

$$\chi(\mathscr{O}_Z, \mathscr{O}_Z) = \sum_\ell (-1)^\ell \chi(E^\ell, \mathscr{O}_Z) = \sum_\ell (-1)^\ell n \cdot \operatorname{rk} E^\ell = 0,$$

proving $\operatorname{ext}^1(\mathscr{O}_Z, \mathscr{O}_Z) = 2n$. □

COROLLARY 1.2.12. *Let X be a smooth surface. Then, the Hilbert–Chow morphism $\sigma_{\mathscr{O}_X, n} \colon \operatorname{Hilb}^n(X) \to \operatorname{Sym}^n(X)$ is a resolution of singularities.*

1.2.6 Torus action on the Hilbert scheme of points

In this chapter we work with the Hilbert scheme

$$\operatorname{Hilb}^n(\mathbb{A}^d).$$

We view it as a fine moduli space of ideals $I \subset \mathbb{C}[x_1, \ldots, x_d]$ of colength n, i.e. such that the quotient algebra $\dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_d]/I$ has dimension n as a complex vector space.

Consider the d-dimensional torus

$$\mathbb{T} = \mathbb{G}_m^d$$

acting on $\mathbb{A}^d$ by

$$t \cdot (a_1, \ldots, a_d) = (t_1 a_1, \ldots, t_d a_d). \tag{1.2.5}$$

Exercise 1.2.13. Show that the action (1.2.5) lifts to a $\mathbb{T}$-action $\mathbb{T} \times \operatorname{Hilb}^n(\mathbb{A}^d) \to \operatorname{Hilb}^n(\mathbb{A}^d)$. Upgrade this replacing $\mathbb{A}^d$ with any smooth toric d-fold (If in need of a hint, open [39, Section 4.2] or [21, Section 9.1]).

Exercise 1.2.14. Show that a $\mathbb{T}$-fixed subscheme $Z \subset \mathbb{A}^d$ is entirely supported at the origin $0 \in \mathbb{A}^d$. (**Hint**: Show that $\operatorname{Supp}(t \cdot [Z]) = t \cdot \operatorname{Supp}(Z)$).

PROPOSITION 1.2.15. *An ideal $I \in \operatorname{Hilb}^n(\mathbb{A}^d)$ is $\mathbb{T}$-fixed if and only if it is a monomial ideal. In particular, the fixed locus $\operatorname{Hilb}^n(\mathbb{A}^d)^{\mathbb{T}}$ is finite.*

Proof. Recall that the character lattice of the torus $\mathbb{T}^* = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ is isomorphic to $\mathbb{Z}^d$, since each character $\mathbb{T} \to \mathbb{G}_m$ is necessarily of the form

$$\chi_{\boldsymbol{m}} \colon (t_1, \ldots, t_d) \longmapsto t_1^{m_1} \cdots t_d^{m_d}$$

for some $\boldsymbol{m}=(m_1,\ldots,m_d)\in\mathbb{Z}^d$. As an initial step, we notice that the geometric action (1.2.5) dualises to a $\mathbb{T}$- action on $\mathbb{C}[x_1,\ldots,x_d]$ via the rule

$$t\cdot f(x_1,\ldots,x_d)=f(t_1^{-1}x_1,\ldots,t_d^{-1}x_d). \tag{1.2.6}$$

This already shows that a monomial ideal is necessarily $\mathbb{T}$-fixed, so it remains to prove the converse.

We next show that the monomials

$$\boldsymbol{x}^{\boldsymbol{m}}=x_1^{m_1}\cdots x_d^{m_d}$$

form an eigenbasis of $\mathbb{C}[x_1,\ldots,x_d]$ as a $\mathbb{T}$-representation. An *eigenvector* of a $\mathbb{T}$-representation V, in this context, is an element $v\in V$ for which there exists a character $\chi\in\mathbb{T}^*$ such that $t\cdot v=\chi(t)v$ for all $t\in\mathbb{T}$. (This χ plays the role of "classical" eigenvalues in linear algebra.) For us, $V=\mathbb{C}[x_1,\ldots,x_d]$. Pick $v=\boldsymbol{x}^{\boldsymbol{m}}$. Then according to (1.2.6) one has

$$t\cdot\boldsymbol{x}^{\boldsymbol{m}}=(t_1^{-1}x_1)^{m_1}\cdots(t_d^{-1}x_d)^{m_d}=t_1^{-m_1}\cdots t_d^{-m_d}(x_1^{m_1}\cdots x_d^{m_d})=\chi_{-\boldsymbol{m}}(t)\boldsymbol{x}^{\boldsymbol{m}}.$$

So each monomial $\boldsymbol{x}^{\boldsymbol{m}}$ is an eigenvector with respect to the weight $\chi_{-\boldsymbol{m}}$. In particular, each corresponds to a different weight, therefore all weight spaces

$$V_{\boldsymbol{m}}=\left\{\,f\in\mathbb{C}[x_1,\ldots,x_d]\,\middle|\,t\cdot f=\chi_{\boldsymbol{m}}(t)f\text{ for all }t\in\mathbb{T}\,\right\}$$

are 1-dimensional $\mathbb{T}$-subrepresentations (each spanned by $\boldsymbol{x}^{\boldsymbol{m}}$) and the action (1.2.6) is diagonalisable by monomials.

Now pick a $\mathbb{T}$-fixed ideal $I\subset V$. In particular, I is a $\mathbb{T}$-subrepresentation. But a $\mathbb{T}$-subrepresentation of a diagonalisable $\mathbb{T}$-representation is again diagonalisable (prove this!), so I has a basis of eigenvectors. But each eigenvector is a multiple of a monomial, thus I is a monomial ideal. □

For $d\in\mathbb{Z}_{\geqslant 1}$, the lattice $\mathbb{N}^d$ is endowed with its standard component-wise poset structure throughout.

Definition 1.2.16. Let $d\geqslant 1$ and $n\geqslant 0$ be integers. A $(d-1)$-dimensional partition of size n is a collection of n points $\lambda=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n\}\subset\mathbb{N}^d$ with the following property: whenever a point $\boldsymbol{y}\in\mathbb{N}^d$ satisfies $\boldsymbol{y}\leqslant\boldsymbol{a}_i$ for all $i=1,\ldots,n$, one has that $\boldsymbol{y}\in\lambda$. We set

$$\mathsf{p}_k(n)=\big|\{\,k\text{-dimensional partitions of }n\,\}\big|.$$

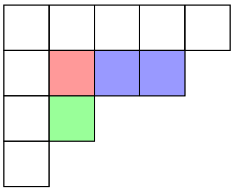
A 1-dimensional partition ($d=2$) is simply called a *partition*. A 2-dimensional partition ($d=3$) is called a *plane partition*.

Example 1.2.17. One has that $\mathsf{p}_0(n)=1$ for all $n\geqslant 0$.

Remark 1.2.18. Partitions correspond bijectively to Young diagrams (also known as Ferrers diagrams), so we will use both terminologies and notations interchangeably. We adopt the English convention for Young diagrams, with the x-axis growing downwards, the y-axis growing rightwards and the box in the corner corresponding to the origin $(0,0)\in\mathbb{A}^2$. See the figures below.

Definition 1.2.19. Let $\lambda\subset\mathbb{N}^2$ be a partition. The *arm-length*, resp. *leg-length* of a box $\square\in\lambda$ is the number of boxes sitting on the right, resp. below $\square$.

Example 1.2.20. In the partition $\lambda = (5,4,2,1)$, the red box $\square = (1,1)$ has arm-length 2 and leg-length 1.



Notation 1.2.21. Set $d = 2$. There are two different (equivalent) ways to represent a partition $\lambda \subset \mathbb{N}^2$, namely

(1) $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0$ with $n = \sum_j \lambda_j$, and

(2) $\lambda = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$ with $n = \sum_i i\alpha_i$ (and some α_i might be 0, in which case they are omitted).

In both cases, we write '$\lambda \vdash n$' to say that λ is a (1-dimensional) partition of n. The notation in (2) means that λ consists of α_i parts of size i for all $i = 1, \ldots, n$. Furthermore:

(i) The quantity
$$||\lambda|| = \sum_i \alpha_i$$
counts the number of rows in the Young diagram attached to λ. It agrees with the number of boxes $\square \in \lambda$ such that $a(\square) = 0$.

(ii) The quantity λ_1 counts the number of columns in the Young diagram attached to λ. It agrees with the number of boxes $\square \in \lambda$ such that $l(\square) = 0$.

(iii) The *automorphism group* of $\lambda \subset \mathbb{N}^2$ is the group $\operatorname{Aut}(\lambda) = \prod_i \mathfrak{S}_{\alpha_i}$, where $\mathfrak{S}_a$ is the symmetric group on a letters.

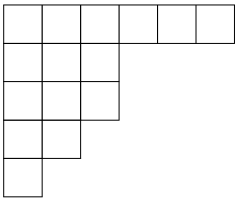


Figure 1.1: the partition $\lambda = (6,3,3,2,1) = (1^1 2^1 3^2 6^1)$ has $||\lambda|| = 1+1+2+1 = 5$, $\lambda_1 = 6$, $\operatorname{Aut}(\lambda) = \mathfrak{S}_2$.

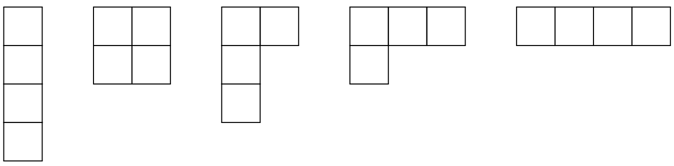


Figure 1.2: The five partitions $\lambda_1, \ldots, \lambda_5$ of 4. According to Notation 1.2.21, $\lambda_1 = (1,1,1,1) = (1^4)$, $\lambda_2 = (2,2) = (2^2)$, $\lambda_3 = (2,1,1) = (1^2 2^1)$, $\lambda_4 = (3,1) = (1^1 3^1)$, $\lambda_5 = (4) = (4^1)$.

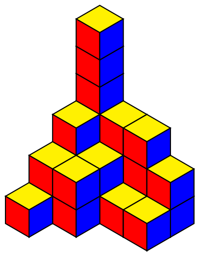


Figure 1.3: A plane partition.

Example 1.2.22. If $d=3$, a *plane partition* of n is the same thing as a way of stacking n boxes in the corner of a room (assuming gravity points in the $(-1,-1,-1)$ direction!).

The following result is a direct consequence of Proposition 1.2.15. The 'proof' is given informally via Figure 1.4.

COROLLARY 1.2.23. *There is a bijective correspondence between $\mathbb{T}$-fixed subschemes $Z \subset \mathbb{A}^d$ of length n and $(d-1)$-dimensional partitions of n.*

1	x_2	x_2^2	x_2^3	x_2^4	x_2^5	x_2^6	x_2^7
x_1	x_1x_2	$x_1x_2^2$	$x_1x_2^3$				
x_1^2	$x_1^2x_2$	$x_1^2x_2^2$					
x_1^3	$x_1^3x_2$	$x_1^3x_2^2$					
x_1^4	$x_1^4x_2$						
x_1^5							
x_1^6							

Figure 1.4: A 1-dimensional partition (Young diagram) draws a staircase and determines (and is determined by) a monomial ideal, in this case $I_\lambda=(x_1^6,x_1^4x_2,x_1^3x_2^2,x_1x_2^3,x_2^7)$. Note that the colength of I_λ is $|\lambda|=17$.

The generating function of the numbers of partitions (for $d=2$) is given by the following formula.

THEOREM 1.2.24 (Euler [18, Chapter 16]). *There is an identity*

$$\sum_{n\geqslant 0}\mathsf{p}_1(n)q^n=\prod_{m\geqslant 1}(1-q^m)^{-1}.$$

Proof. By contemplation. For instance, say we want to compute $\mathsf{p}_1(4)$, that we know equals 5 (cf. Figure 1.2). Then expanding

$$\prod_{m\geqslant 1}(1-q^m)^{-1}=(1+q^1+q^{1+1}+q^{1+1+1}+q^{1+1+1+1}+\cdots)$$
$$\cdot(1+q^2+q^{2+2}+q^{2+2+2}+\cdots)\cdot(1+q^3+q^{3+3}+\cdots)\cdot(1+q^4+q^{4+4}+\cdots)\cdots$$

we see that to compute the coefficient of q^4 we have to sum the coefficients of

$$\begin{aligned}&q^4\\&q^1\cdot q^3\\&q^{1+1}\cdot q^2\\&q^{1+1+1+1}\\&q^{2+2}.\end{aligned}$$

These clearly correspond to partitions of 4. □

Definition 1.2.25 (MacMahon function)**.** The *MacMahon function* is the infinite product

$$\mathsf{M}(q)=\prod_{m\geqslant 1}(1-q^m)^{-m}.$$

The generating function of the numbers of plane partitions (for $d=3$) is given by the following formula.

THEOREM 1.2.26 (MacMahon [34]). *There is an identity*

$$\sum_{n\geqslant 0}\mathsf{p}_2(n)q^n=\mathsf{M}(q).$$

1.2.7 The character of the tangent space

The goal of this section is to compute the character of the tangent representation at a torus fixed point (monomial ideal) of $\mathsf{Hilb}^n(\mathbb{A}^d)$.

Consider the torus $\mathbb{T}=\mathbb{G}_m^d$. Let $K_0^{\mathbb{T}}(\mathsf{pt})$ be the K-group of the abelian category of $\mathbb{T}$-representations. Taking a representation to its weight decomposition is a ring isomorphism

$$\chi\colon K_0^{\mathbb{T}}(\mathsf{pt})\xrightarrow{\sim}\mathbb{Z}[t_1^{\pm},\ldots,t_d^{\pm}],$$

so we will identify the two rings without further mention. In order to allow infinite-dimensional representations, and also (just to be safe) rational coefficients, we define

$$\widehat{K}_0^{\mathbb{T}}(\mathsf{pt})=\mathbb{Q}(\!(t_1,\ldots,t_d)\!).$$

Define, for F,G two equivariant coherent sheaves on $\mathbb{A}^d$, the representations

$$\chi(F,G)=\sum_{i\geqslant 0}(-1)^i\operatorname{Ext}^i(F,G)\in\widehat{K}_0^{\mathbb{T}}(\mathsf{pt}),$$

and set $\chi(G)=\chi(\mathscr{O}_{\mathbb{A}^d},G)$. These representations satisfy the usual relation

$$\chi(F,G)=\chi(F^*\otimes G). \tag{1.2.7}$$

Example 1.2.27. The canonical line bundle $K_{\mathbb{A}^d}$ is equivariantly nontrivial, as

$$K_{\mathbb{A}^d}=\mathscr{O}_{\mathbb{A}^d}\otimes(t_1\cdots t_d)^{-1}.$$

Indeed, the sheaf of Kähler differentials splits as $\Omega_{\mathbb{A}^d}=\mathscr{O}_{\mathbb{A}^d}\mathrm{d}x_1\oplus\cdots\oplus\mathscr{O}_{\mathbb{A}^d}\mathrm{d}x_d$ and x_i carries a -1 weight. This also gives the relation

$$\chi(T_0\mathbb{A}^d)=t_1+\cdots+t_d. \tag{1.2.8}$$

Example 1.2.28. Since the action on coordinates is given by $t_i \cdot x_i = t_i^{-1} x_i$, and monomials span $\mathbb{C}[x_1, \ldots, x_d]$, we have

$$\chi(\mathcal{O}_{\mathbb{A}^d}) = \sum_{m_1,\ldots,m_d \geqslant 0} t_1^{-m_1} \cdots t_d^{-m_d} = \prod_{1 \leqslant i \leqslant d} \frac{1}{(1-t_i^{-1})}.$$

Denote by $\overline{(\cdot)}$ the involution on $K_0^{\mathbb{T}}(\mathrm{pt})$ sending t_i to t_i^{-1}. Then one has

$$\frac{1}{\overline{\chi(\mathcal{O}_{\mathbb{A}^d})}} = (1-t_1)\cdots(1-t_d) = (-1)^d (t_1 \cdots t_d)(1-t_1^{-1})\cdots(1-t_d^{-1}) = \frac{(-1)^d(t_1\cdots t_d)}{\chi(\mathcal{O}_{\mathbb{A}^d})},$$

which is equivalent to the relation

$$\chi(\mathcal{O}_{\mathbb{A}^d}) = (-1)^d (t_1 \cdots t_d)\overline{\chi(\mathcal{O}_{\mathbb{A}^d})}. \tag{1.2.9}$$

Definition 1.2.29. The tautological (rank n) bundle over $\mathsf{Hilb}^n(\mathbb{A}^d)$ is the pushdown

$$\mathcal{V} = q_* \mathcal{O}_{\mathcal{Z}}$$

of the structure sheaf of the universal subscheme $\mathcal{Z} \subset \mathbb{A}^d \times \mathsf{Hilb}^n(\mathbb{A}^d)$ along the projection map $q\colon \mathbb{A}^d \times \mathsf{Hilb}^n(\mathbb{A}^d) \to \mathsf{Hilb}^n(\mathbb{A}^d)$. The restriction $\mathcal{V}_\lambda = \mathcal{V}|_{\mathscr{I}_\lambda}$ to a point $[\mathscr{I}_\lambda] \in \mathsf{Hilb}^n(\mathbb{A}^d)$ corresponding to a partition $\lambda \subset \mathbb{N}^d$ is naturally a finite dimensional $\mathbb{T}$-representation (of rank n), i.e. an element of $K^0_{\mathbb{T}}(\mathrm{pt})$.

LEMMA 1.2.30. *Let S be a nonsingular quasiprojective toric surface. Lift the natural action of $\mathbb{T} = \mathbb{G}_m^2$ on S to $\mathsf{Hilb}^n(S)$. Then, at a $\mathbb{T}$-fixed point $\mathscr{I}_Z \in \mathsf{Hilb}^n(S)$, one can write the tangent space as*

$$T_{\mathscr{I}_Z} \mathsf{Hilb}^n(S) = \chi(\mathcal{O}_S) - \chi(\mathscr{I}_Z, \mathscr{I}_Z) \in K_0^{\mathbb{T}}(\mathrm{pt}).$$

Proof. Let us compute

$$\begin{aligned}
\chi(\mathcal{O}_S) - \chi(\mathscr{I}_Z,\mathscr{I}_Z) &= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S - \mathcal{O}_Z, \mathcal{O}_S - \mathcal{O}_Z)\\
&= \chi(\mathcal{O}_S,\mathcal{O}_Z) + \chi(\mathcal{O}_Z,\mathcal{O}_S) - \chi(\mathcal{O}_Z,\mathcal{O}_Z)\\
&= \chi(\mathscr{I}_Z,\mathcal{O}_Z) + \chi(\mathcal{O}_Z,\mathcal{O}_Z) + \chi(\mathcal{O}_Z,\mathcal{O}_S) - \chi(\mathcal{O}_Z,\mathcal{O}_Z)\\
&= \chi(\mathscr{I}_Z,\mathcal{O}_Z) + \chi(\mathcal{O}_Z,\mathcal{O}_S).
\end{aligned}$$

Applying $\mathrm{Hom}(-, \mathcal{O}_Z)$ to the ideal sheaf exact sequence defining $Z \subset S$, we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{O}_Z,\mathcal{O}_Z) \arrow[r, "\sim"] & \mathrm{Hom}(\mathcal{O}_S,\mathcal{O}_Z) \arrow[r, "0"] & \mathrm{Hom}(\mathscr{I}_Z,\mathcal{O}_Z) \arrow[dll, out=-10, in=170] \\
& \mathrm{Ext}^1(\mathcal{O}_Z,\mathcal{O}_Z) \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^1(\mathscr{I}_Z,\mathcal{O}_Z) \arrow[dll, out=-10, in=170] \\
& \mathrm{Ext}^2(\mathcal{O}_Z,\mathcal{O}_Z) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

where we have used the vanishings $\mathrm{Ext}^{>0}(\mathcal{O},\mathcal{O}_Z) = 0$ and $\mathrm{Ext}^2(\mathscr{I}_Z,\mathcal{O}_Z) = \mathrm{Hom}(\mathcal{O}_Z, \mathscr{I}_Z \otimes \omega_S)^\vee = 0$. We also have

$$\mathrm{Ext}^1(\mathscr{I}_Z,\mathcal{O}_Z) \xrightarrow{\sim} \mathrm{Ext}^2(\mathcal{O}_Z,\mathcal{O}_Z) \cong \mathrm{Hom}(\mathcal{O}_Z,\mathcal{O}_Z)^\vee \cong \mathrm{Hom}(\mathcal{O}_S,\mathcal{O}_Z)^\vee$$

to be read off from the long exact sequence. Therefore

$$\begin{aligned}
\chi(\mathcal{O}_S) - \chi(\mathscr{I}_Z,\mathscr{I}_Z) &= \chi(\mathscr{I}_Z,\mathcal{O}_Z) + \chi(\mathcal{O}_Z,\mathcal{O}_S)\\
&= T_{\mathscr{I}_Z}\mathsf{Hilb}^n(S) - \mathrm{Ext}^1(\mathscr{I}_Z,\mathcal{O}_Z) + \mathrm{Ext}^2(\mathcal{O}_Z,\mathcal{O}_S)\\
&= T_{\mathscr{I}_Z}\mathsf{Hilb}^n(S) - \mathrm{Hom}(\mathcal{O}_S,\mathcal{O}_Z)^\vee + \mathrm{Hom}(\mathcal{O}_S,\mathcal{O}_Z)^\vee\\
&= T_{\mathscr{I}_Z}\mathsf{Hilb}^n(S).
\end{aligned}$$

□

LEMMA 1.2.31. *Let F be an equivariant coherent sheaf on $\mathbb{A}^d$. Then one has the relation*

$$\chi(F) = \chi(F \otimes \mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^d}). \tag{1.2.10}$$

Proof. Set $R = \mathbb{C}[x_1,\ldots,x_d]$ and $M = T_0^*\mathbb{A}^d$, which as a representation is just $t_1^{-1}+\cdots+t_d^{-1}$ by (1.2.8). Note that $0 \in \mathbb{A}^d$ is the unique torus fixed point. The structure sheaf $\mathscr{O}_0 = R/\mathfrak{m}$ has a torus equivariant Koszul resolution

$$0 \longrightarrow R \otimes \bigwedge^d M \longrightarrow R \otimes \bigwedge^{d-1} M \longrightarrow \cdots \longrightarrow R \otimes \bigwedge^1 M \longrightarrow R \longrightarrow \mathscr{O}_0 \longrightarrow 0.$$

Out of this, we get

$$\chi(\mathscr{O}_0) = \chi(\wedge^\bullet M) = \sum_{1\leqslant i\leqslant d} (-1)^i \chi(\wedge^i M) = \prod_{1\leqslant i\leqslant d}(1-t_i^{-1}) = \frac{1}{\chi(\mathscr{O}_{\mathbb{A}^d})}.$$

Now, $\chi(F\otimes \mathscr{O}_0) = \chi(F)\chi(\mathscr{O}_0)$ gives the result. □

The character of the tangent space at a fixed point of the Hilbert scheme (i.e. its decomposition into weight spaces) can be computed combining Lemma 1.2.30 with the following formula.

LEMMA 1.2.32. *Let F, G two equivariant coherent sheaves on $\mathbb{A}^d$. Denote by $\overline{(\cdot)}$ the involution on $K_0^{\mathbb{T}}(\mathrm{pt})$ sending t_i to t_i^{-1}. Then one has the relation*

$$\chi(F,G) = \frac{\overline{\chi(F)}\chi(G)}{\overline{\chi(\mathscr{O}_{\mathbb{A}^d})}}. \tag{1.2.11}$$

Proof. Since stalks and tensor products commute, we have

$$\begin{aligned} \chi((F\otimes F')\otimes \mathscr{O}_0) &= \chi((F\otimes \mathscr{O}_0)\otimes(F'\otimes\mathscr{O}_0)) \\ &= \chi(F\otimes\mathscr{O}_0)\chi(F'\otimes\mathscr{O}_0). \end{aligned} \tag{1.2.12}$$

Also note that Serre duality $(-1)^d\overline{\chi(F',F)} = \chi(F, F'\otimes K_{\mathbb{A}^d})$ specialises (when F' is trivial) to

$$(-1)^d\overline{\chi(F)} = \chi(F, K_{\mathbb{A}^g}) = \chi(F^*\otimes K_{\mathbb{A}^g}) = \chi(F^*)(t_1\cdots t_d)^{-1}. \tag{1.2.13}$$

So we can compute

$$\begin{aligned} \chi(F,G) &= \chi(F^*\otimes G) && \text{by (1.2.7)} \\ &= \chi((F^*\otimes G)\otimes\mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^g}) && \text{by (1.2.10)} \\ &= \chi(F^*\otimes\mathscr{O}_0)\chi(G\otimes\mathscr{O}_0)\chi(\mathscr{O}_{\mathbb{A}^g}) && \text{by (1.2.12)} \\ &= \chi(F^*)\chi(G\otimes\mathscr{O}_0) && \text{by (1.2.10)} \\ &= (-1)^d(t_1\cdots t_d)\overline{\chi(F)}\chi(G\otimes\mathscr{O}_0) && \text{by (1.2.13)} \\ &= \frac{\chi(\mathscr{O}_{\mathbb{A}^d})}{\overline{\chi(\mathscr{O}_{\mathbb{A}^d})}}\overline{\chi(F)}\chi(G\otimes\mathscr{O}_0) && \text{by (1.2.9)} \\ &= \frac{\overline{\chi(F)}\chi(G)}{\overline{\chi(\mathscr{O}_{\mathbb{A}^d})}} && \text{by (1.2.10)} \end{aligned}$$

as required. □

Let us go back to the case $d=2$. The torus action is conceived so that one has

$$\begin{aligned} \chi(\mathscr{O}_{\mathbb{A}^2}) &= \sum_{\square\in\mathbb{Z}^2_{\geqslant 0}} t^{-\square} = \frac{1}{(1-t_1^{-1})(1-t_2^{-1})} \\ \overline{\chi(\mathscr{O}_{\mathbb{A}^2})} &= \frac{1}{(1-t_1)(1-t_2)} \\ \overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} &= (1-t_1)(1-t_2) = (1-t_1^{-1})(1-t_2^{-1})t_1t_2. \end{aligned}$$

If $Z \subset \mathbb{A}^2$ defines a torus fixed point, it is determined by a monomial ideal $\mathscr{I}_\lambda \subset \mathbb{C}[x_1, x_2]$, corresponding to a partition (Young diagram) $\lambda \vdash n$. The character associated to the $\mathbb{T}$-module $\mathscr{O}_Z$ (the restriction of the tautological bundle to the point determined by Z) is

$$\mathcal{V}_\lambda = \chi(\mathscr{O}_Z) = \sum_{\square \in \lambda} t^{-\square} = \sum_{(i,j)\in\lambda} t_1^{-i} t_2^{-j}.$$

It follows from Formula (1.2.11) that one can compute

$$\begin{aligned} T_{\mathscr{I}_\lambda} \mathsf{Hilb}^n(\mathbb{A}^2) &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \chi(\mathscr{I}_\lambda, \mathscr{I}_\lambda) \\ &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z, \mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z) \\ &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \overline{\chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z)}\chi(\mathscr{O}_{\mathbb{A}^2} - \mathscr{O}_Z)\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} \\ &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \left[\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}\chi(\mathscr{O}_{\mathbb{A}^2}) - \overline{\chi(\mathscr{O}_{\mathbb{A}^2})}\mathcal{V}_\lambda - \overline{\mathcal{V}}_\lambda \chi(\mathscr{O}_{\mathbb{A}^2}) + \overline{\mathcal{V}}_\lambda \mathcal{V}_\lambda\right]\overline{\chi(\mathscr{O}_{\mathbb{A}^2})}^{-1} \\ &= \chi(\mathscr{O}_{\mathbb{A}^2}) - \left[\chi(\mathscr{O}_{\mathbb{A}^2}) - \mathcal{V}_\lambda - \overline{\mathcal{V}}_\lambda t_1 t_2 + \overline{\mathcal{V}}_\lambda \mathcal{V}_\lambda (1-t_1)(1-t_2)\right] \\ &= \mathcal{V}_\lambda + \overline{\mathcal{V}}_\lambda t_1 t_2 - \overline{\mathcal{V}}_\lambda \mathcal{V}_\lambda (1-t_1)(1-t_2). \end{aligned} \tag{1.2.14}$$

Exercise 1.2.33. Let $\lambda \vdash n$ be a partition. Prove[1] that

$$T_{\mathscr{I}_\lambda} \mathsf{Hilb}^n(\mathbb{A}^2) = \sum_{\square\in\lambda} t_1^{-l(\square)} t_2^{a(\square)+1} + t_1^{l(\square)+1} t_2^{-a(\square)}. \tag{1.2.15}$$

Example 1.2.34. Let us confirm the conclusion of Exercise 1.2.33 for $n \leqslant 3$.

<u>$n = 1$.</u> Equation (1.2.14) gives

$$T_\square \mathsf{Hilb}^1(\mathbb{A}^2) = 1 + t_1 t_2 - (1-t_1)(1-t_2) = t_1 + t_2.$$

On the other hand, Equation (1.2.15) gives

$$T_\square \mathsf{Hilb}^1(\mathbb{A}^2) = t_1^0 t_2^1 + t_1^1 t_2^0 = t_1 + t_2,$$

as desired.

<u>$n = 2$.</u> Here we have two partitions to consider. We have

$$\mathcal{V}_{\substack{\square\\\square}} = 1 + t_1^{-1}, \qquad \mathcal{V}_{\square\square} = 1 + t_2^{-1}.$$

So Equation (1.2.14) gives

$$\begin{aligned} T_{\substack{\square\\\square}} \mathsf{Hilb}^2(\mathbb{A}^2) &= 1 + t_1^{-1} + (1+t_1)t_1 t_2 - (1+t_1)(1+t_1^{-1})(1-t_1)(1-t_2) \\ &= t_1 + t_2 + t_1^2 + t_1^{-1} t_2 \\ T_{\square\square} \mathsf{Hilb}^2(\mathbb{A}^2) &= 1 + t_2^{-1} + (1+t_2)t_1 t_2 - (1+t_2)(1+t_2^{-1})(1-t_1)(1-t_2) \\ &= t_1 + t_2 + t_2^2 + t_1 t_2^{-1}. \end{aligned}$$

On the other hand, Equation (1.2.15) gives

$$\begin{aligned} T_{\substack{\square\\\square}} \mathsf{Hilb}^2(\mathbb{A}^2) &= (t_1^{-1} t_2 + t_1^2 t_2^0) + (t_1^0 t_2 + t_1^1 t_2^0) \\ &= t_1 + t_2 + t_1^2 + t_1^{-1} t_2 \\ T_{\square\square} \mathsf{Hilb}^2(\mathbb{A}^2) &= (t_1^0 t_2^2 + t_1 t_2^{-1}) + (t_1^0 t_2 + t_1^1 t_2^0) \\ &= t_1 + t_2 + t_2^2 + t_1 t_2^{-1}. \end{aligned}$$

[1] This has been known for a while but I do not know an "old" reference; a recent proof is given in [36, Lemma 2.1].

<u>$n=3$.</u> Here we have three partitions to consider. We have

$$\mathcal{V}_{\yng(3)} = 1+t_2^{-1}+t_2^{-2}, \qquad \mathcal{V}_{\yng(1,1,1)} = 1+t_1^{-1}+t_1^{-2}, \qquad \mathcal{V}_{\yng(2,1)} = 1+t_1^{-1}+t_2^{-1}.$$

So Equation (1.2.14) gives

$$\begin{aligned}
T_{\yng(3)} &= 1+t_2^{-1}+t_2^{-2}+(1+t_2+t_2^2)t_1t_2-(1+t_2+t_2^2)(1+t_2^{-1}+t_2^{-2})(1-t_1)(1-t_2)\\
&= t_1+t_2+t_2^2+t_2^3+t_1t_2^{-1}+t_1t_2^{-2}\\
T_{\yng(1,1,1)} &= 1+t_1^{-1}+t_1^{-2}+(1+t_1+t_1^2)t_1t_2-(1+t_1+t_1^2)(1+t_1^{-1}+t_1^{-2})(1-t_1)(1-t_2)\\
&= t_1+t_2+t_1^2+t_1^3+t_1^{-1}t_2+t_1^{-2}t_2\\
T_{\yng(2,1)} &= 1+t_1^{-1}+t_2^{-1}+(1+t_1+t_2)t_1t_2-(1+t_1+t_2)(1+t_1^{-1}+t_2^{-1})(1-t_1)(1-t_2)\\
&= 2t_1+2t_2+t_1^2t_2^{-1}+t_1^{-1}t_2^2.
\end{aligned}$$

On the other hand, Equation (1.2.15) gives

$$\begin{aligned}
T_{\yng(3)} &= (t_1^0t_2^3+t_1t_2^{-2})+(t_1^0t_2^2+t_1t_2^{-1})+(t_1^0t_2+t_1t_2^0)\\
&= t_1+t_2+t_2^2+t_2^3+t_1t_2^{-1}+t_1t_2^{-2}\\
T_{\yng(1,1,1)} &= (t_1^{-2}t_2+t_1^3t_2^0)+(t_1^{-1}t_2+t_1^2t_2^0)+(t_1^0t_2+t_1t_2^0)\\
&= t_1+t_2+t_1^2+t_1^3+t_1^{-1}t_2+t_1^{-2}t_2\\
T_{\yng(2,1)} &= (t_1^{-1}t_2^2+t_1^2t_2^{-1})+(t_1^0t_2+t_1t_2^0)+(t_1^0t_2+t_1t_2^0)\\
&= 2t_1+2t_2+t_1^2t_2^{-1}+t_1^{-1}t_2^2.
\end{aligned} \tag{1.2.16}$$

2 | Motivic techniques

2.1 Grothendieck ring of varieties and attached gadgets

2.1.1 Absolute setup: motivic measures and zeta functions

The Grothendieck ring of complex varieties $K_0(\mathsf{Var}_{\mathbb{C}})$ is the free abelian group generated by isomorphism classes $[X]$ of $\mathbb{C}$-varieties, modulo the *scissor relations*, namely the relations

$$[X] = [Y] + [X \setminus Y]$$

whenever $Y \subset X$ is a closed subvariety. The group structure agrees, on generators, with the disjoint union of varieties and has neutral element $0 = [\emptyset]$. The fibre product defines a ring structure on $K_0(\mathsf{Var}_{\mathbb{C}})$, with neutral element $1 = [\operatorname{Spec}\mathbb{C}]$. Every constructible subset $Z \subset X$ of a variety X has a well-defined motivic class $[Z]$, independent upon the decomposition of Z into locally closed subsets. One could also perform a similar construction with schemes (or algebraic spaces) over $\mathbb{C}$ instead of varieties, but the resulting Grothendieck rings would come out isomorphic to $K_0(\mathsf{Var}_{\mathbb{C}})$. By the scissor relations, the motive of a scheme agrees with the motive of its reduction, so $[-]$ is insensitive to scheme structures.

The classes $[X]$ of honest $\mathbb{C}$-varieties in $K_0(\mathsf{Var}_{\mathbb{C}})$ are called *effective*. A key example is the *Lefschetz motive*

$$\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathsf{Var}_{\mathbb{C}}).$$

The ring $K_0(\mathsf{Var}_{\mathbb{C}})$ has the following universal property. Suppose R is a ring and $w(-)$ is an R-valued invariant of algebraic varieties, such that

- $w(\mathsf{pt}) = 1$,
- $w(\emptyset) = 0$,
- $w(X \times Y) = w(X)w(Y)$ for every two varieties X and Y, and
- $w(X) = w(Y) + w(X \setminus Y)$ for every variety X and closed subvariety $Y \subset X$.

Then there is precisely one ring homomorphism $w\colon K_0(\mathsf{Var}_{\mathbb{C}}) \to R$ sending an effective class $[X]$ to $w(X) \in R$. Ring homomorphisms out of $K_0(\mathsf{Var}_{\mathbb{C}})$ are called *motivic measures*, or generalised Euler characteristics, or realisations [13, 33]. The key examples are the following. If Y is a complex variety, one can consider

(i) The *Hodge characteristic*

$$\chi_{\text{Hodge}}(Y) = \sum_{i \geqslant 0} (-1)^i [\mathrm{H}^i_c(Y, \mathbb{Q})] \in K_0(\mathsf{HS}),$$

where HS is the abelian category of Hodge structures, and the vector space $\mathrm{H}^i_c(Y,\mathbb{Q})$ is equipped with Deligne's mixed Hodge structure [11, 12].

(ii) The *Hodge–Deligne polynomial*

$$\mathsf{E}(Y;u,v)=\sum_{p,q,i}(-1)^i h^{p,q}(\mathrm{H}^i_c(Y,\mathbb{Q}))u^p v^q\in\mathbb{Z}[u,v],$$

where $h^{p,q}(\mathrm{H}^i_c(Y,\mathbb{Q}))$ is the dimension of the (p,q)-component of the mixed Hodge structure $\mathrm{H}^i_c(Y,\mathbb{Q})$.

(iii) The *weight polynomial* $(u,v\mapsto z^{1/2})$

$$\mathsf{w}(Y,z^{1/2})=\sum_{p,q,i}(-1)^i h^{p,q}(\mathrm{H}^i_c(Y,\mathbb{Q}))z^{(p+q)/2}\in\mathbb{Z}[z^{1/2}],$$

which coincides with the signed *Poincaré polynomial*

$$\mathsf{p}(Y,-z^{1/2})=\sum_{i\geqslant 0}\dim_{\mathbb{Q}}\mathrm{H}^i(Y,\mathbb{Q})(-z^{1/2})^i$$

if Y is smooth and projective.

(iv) The *compactly supported Euler characteristic* $(z^{1/2}\mapsto 1)$

$$\chi_c(Y)=\mathsf{w}(Y,1)=\sum_{i\geqslant 0}(-1)^i\dim_{\mathbb{Q}}\mathrm{H}^i_c(Y,\mathbb{Q})\in\mathbb{Z},$$

which can be shown to equal the topological Euler characteristic $\chi(Y)$ [23, p. 95 and pp. 141–142].

Remark 2.1.1. It is possible to obtain homomorphisms

$$K_0(\mathsf{Var}_{\mathbb{C}})\to K_0(\mathsf{MHS})\to\mathbb{Z}[u,v]\to\mathbb{Z}[z^{1/2}]\to\mathbb{Z}$$

with slightly different specialisations, but with our choice we ensured that all specialisations are homomorphisms of pre-λ-rings, just as in [10, Sec. 1.4]. Note that these homomorphisms, the way they have been defined, send

$$\mathbb{L}\mapsto[\mathrm{H}^2_c(\mathbb{A}^1,\mathbb{Q})]\mapsto uv\mapsto z\mapsto 1.$$

The key features of $K_0(\mathsf{Var}_{\mathbb{C}})$ to keep in mind, which will be used constantly throughout, are the following:

(a) If $X\to B$ is a bijective morphism, then $[X]=[B]$ in $K_0(\mathsf{Var}_{\mathbb{C}})$.

(b) If $X\to B$ is a Zariski locally trivial fibration with fibre F, then $[X]=[B][F]$ in $K_0(\mathsf{Var}_{\mathbb{C}})$.

Exercise 2.1.2. Prove (a) and (b).

As a special case of (a), consider a scheme X along with locally closed subschemes $Z_1,\ldots,Z_r$ such that the disjoint union of the locally closed immersions $Z_1\amalg\cdots\amalg Z_r\to X$ is a bijective morphism (in this case we say that $Z_1,\ldots,Z_r$ form a *stratification* of X, and we write $X=\coprod_i Z_i$ with a slight abuse of notation). One then has an identity $[X]=\sum_{1\leqslant i\leqslant r}[Z_i]$ in $K_0(\mathsf{Var}_{\mathbb{C}})$. For instance, $\mathbb{P}^n$ has the same class as the scheme $\mathrm{pt}\amalg\mathbb{A}^1\amalg\cdots\amalg\mathbb{A}^n$, namely $1+\mathbb{L}+\cdots+\mathbb{L}^n$.

Example 2.1.3. The class of the Grassmannian $\mathsf{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$ is the '$\mathbb{L}$-binomial coefficient', namely

$$[\mathsf{Gr}(k,n)] = \frac{[n]_{\mathbb{L}}!}{[k]_{\mathbb{L}}![n-k]_{\mathbb{L}}!}, \qquad [a]_{\mathbb{L}}! = \prod_{k=1}^{a}\left(\mathbb{L}^k - 1\right). \tag{2.1.1}$$

Throughout we adopt the conventions $[\mathsf{Gr}(k,n)] = 0 = [\mathbb{P}^e]$ if $k > n$ or $e < 0$.

Every complex variety X has an associated *zeta function* $\zeta_X(t)$, introduced by Kapranov [32]. It is defined as

$$\zeta_X(t) = \sum_{d \geqslant 0} [\mathsf{Sym}^d(X)]t^d \in 1 + tK_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!]. \tag{2.1.2}$$

It satisfies, for every integer $n \geqslant 0$, the identities

$$\zeta_{\mathbb{A}^n \times X}(t) = \zeta_X(\mathbb{L}^n t), \qquad \zeta_{X \amalg Y}(t) = \zeta_X(t)\zeta_Y(t),$$

from which one deduces

$$\zeta_{\mathbb{A}^n}(t) = \frac{1}{1-\mathbb{L}^n t}, \qquad \zeta_{\mathbb{P}^n}(t) = \prod_{i=0}^{n} \frac{1}{1-\mathbb{L}^i t}.$$

2.1.2 Relative setup: main structures and operations

Relative motives

Fix a scheme S locally of finite type over $\mathbb{C}$. The Grothendieck ring of S-varieties

$$K_0(\mathsf{Var}_S)$$

is the free abelian group generated by isomorphism classes $[X \to S]$ of S-varieties modulo the *scissor relations*, namely the identities

$$\left[X \xrightarrow{f} S\right] = \left[Y \xrightarrow{f|_Y} S\right] + \left[X \setminus Y \xrightarrow{f|_{X\setminus Y}} S\right]$$

imposed whenever $Y \subset X$ is a closed S-subvariety of X. The ring structure is given on generators by fibre product over S,

$$[X \to S] \cdot [Y \to S] = [X \times_S Y \to S]. \tag{2.1.3}$$

The neutral element for the sum is $[\emptyset \to S]$, whereas for the product it is the class $[\mathrm{id}\colon S \to S]$ of the identity. Clearly, the case $S = \operatorname{Spec}\mathbb{C}$ recovers the ring $K_0(\mathsf{Var}_{\mathbb{C}})$ of Section 2.1.1. The element

$$\mathbb{L} = [\mathbb{A}^1 \times_{\mathbb{C}} S \to S] \in K_0(\mathsf{Var}_S)$$

is again called the *Lefschetz motive* (over S). If S' is another complex scheme, there is an external product

$$K_0(\mathsf{Var}_S) \times K_0(\mathsf{Var}_{S'}) \xrightarrow{\boxtimes} K_0(\mathsf{Var}_{S \times S'}) \tag{2.1.4}$$

defined on generators by sending $([f\colon X \to S], [g\colon X' \to S']) \mapsto [f \times g\colon X \times X' \to S \times S']$.

A morphism $f\colon S \to T$ induces a ring homomorphism $f^*\colon K_0(\mathsf{Var}_T) \to K_0(\mathsf{Var}_S)$ by base change and a $K_0(\mathsf{Var}_T)$-linear map $f_!\colon K_0(\mathsf{Var}_S) \to K_0(\mathsf{Var}_T)$ defined on generators by composition with f.

Definition 2.1.4. We denote by $S_0(\mathsf{Var}_S)$ the semigroup of *effective* motives, i.e. the semigroup generated by isomorphism classes $[X \to S]$ of complex quasiprojective S-varieties modulo the scissor relations. The product (2.1.3) turns $S_0(\mathsf{Var}_S)$ into a semiring. There is a natural semiring map $S_0(\mathsf{Var}_S) \to K_0(\mathsf{Var}_S)$, and we say that a motivic class $\alpha \in K_0(\mathsf{Var}_S)$ is effective if it lies in the image of this map.

Equivariant motives and the quotient map

Recall that if S is a scheme with a *good* action by a finite group G (i.e. an action such that every point of S has an affine G-invariant open neighborhood), the quotient S/G exists as a scheme. For instance, finite group actions on quasiprojective varieties are good.

Definition 2.1.5. Let G be a finite group, S a scheme with good G-action. We denote by $\widetilde{K}_0^G(\mathsf{Var}_S)$ the free abelian group generated by isomorphism classes $[X \to S]$ of G-equivariant S-varieties with good action, modulo the G-equivariant scissor relations. We denote by $K_0^G(\mathsf{Var}_S)$ the quotient of $\widetilde{K}_0^G(\mathsf{Var}_S)$ by the relations

$$[V \to X \to S] = [\mathbb{A}_X^r \to X \to S],$$

where $V \to X$ is a G-equivariant vector bundle of rank r over a G-equivariant S-variety $X \to S$.

Notation 2.1.6. Sometimes we write $[X \to S, G]$ to remember the group action.

There is a natural ring structure on $\widetilde{K}_0^G(\mathsf{Var}_S)$, where the product of two classes $[X \to S]$ and $[Y \to S]$ is given by taking the diagonal action on $X \times_S Y$. The structures f^*, $f_!$ and $\boxtimes$ naturally extend to the equivariant setting, along with their basic compatibilities. For instance, if $f : S \to T$ (resp. $g : S' \to T'$) is a G-equivariant (resp. G'-equivariant) map, and α, β are equivariant motives over S, S', then

$$(f \times g)_!(\alpha \boxtimes \beta) = f_!\alpha \boxtimes g_!\beta \tag{2.1.5}$$

in the $(G \times G')$-equivariant K-group over $T \times T'$.

Recall that, since the G-action on S is good, S/G exists as a scheme. One can define a $K_0(\mathsf{Var}_{S/G})$-linear map (cf. [10, Lemma 1.5])

$$\widetilde{K}_0^G(\mathsf{Var}_S) \xrightarrow{\pi_G} K_0(\mathsf{Var}_{S/G}) \tag{2.1.6}$$

given on generators by taking the orbit space,

$$\pi_G[X \to S] = [X/G \to S/G].$$

This map does not always extend to $K_0^G(\mathsf{Var}_S)$. It does when G acts freely on S, by [6, Lemma 3.2].

Lambda ring structures

Let $n > 0$ be an integer, and let $\mathfrak{S}_n$ be the symmetric group of n elements. By [10, Lemma 1.6], namely the relative version of [2, Lemma 2.4], there exist "n-th power" maps

$$K_0(\mathsf{Var}_S) \xrightarrow{(\cdot)^{\otimes n}} \widetilde{K}_0^{\mathfrak{S}_n}(\mathsf{Var}_{S^n}) \tag{2.1.7}$$

where $S^n = S \times \cdots \times S$ is endowed with the natural $\mathfrak{S}_n$-action. The power map takes $[f : X \to S]$ to the class of the equivariant function $f^n : X^n \to S^n$. For $A \in K_0(\mathsf{Var}_S)$, consider the classes

$$\pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathsf{Var}_{S^n/\mathfrak{S}_n}) = K_0(\mathsf{Var}_{\mathsf{Sym}^n(S)}).$$

The ring $K_0(\mathsf{Var}_{\mathbb{C}})$ is a *lambda ring*, in which the lambda ring operations are defined by

$$A \mapsto \sigma^n(A) = \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathsf{Var}_{\mathbb{C}})$$

for effective classes $A \in K_0(\mathsf{Var}_{\mathbb{C}})$, and then taking the unique extension to a lambda ring structure on $K_0(\mathsf{Var}_{\mathbb{C}})$, determined by the relation

$$\sum_{i=0}^{n} \sigma^i([X]-[Y])\sigma^{n-i}[Y] = \sigma^n[X]. \tag{2.1.8}$$

If S comes with a commutative associative map $\nu\colon S \times S \to S$, we likewise define

$$\sigma^n_\nu(A) = \overline{\nu}_! \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathsf{Var}_S)$$

on effective classes $A = [X \to S]$, where $\overline{\nu}$ is the map $\mathsf{Sym}^n(S) = S^n/\mathfrak{S}_n \to S$. One then uses the analogue of the relation (2.1.8) to find a unique set of lambda ring operators σ^n_ν restricting to the previous identity on effective motives.

As a special case, one can consider $(S,\nu) = (\mathbb{N},+)$, viewed as a symmetric monoid in the category of schemes. We obtain lambda operations $\sigma^n = \sigma^n_+$ on $K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!]$ via the isomorphism

$$K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!] \xrightarrow{\sim} K_0(\mathsf{Var}_{\mathbb{N}}) \tag{2.1.9}$$

defined by sending $\sum_{n\geqslant 0}[Y_n]t^n \mapsto \left[\coprod_{n\in\mathbb{N}} Y_n \to \mathbb{N}\right]$.

2.1.3 Power structures

The main references for power structures are [24, 25].

Definition 2.1.7 ([24])**.** A *power structure* on a (semi)ring R is a map

$$(1+tR[\![t]\!]) \times R \to 1 + tR[\![t]\!]$$
$$(A(t), m) \mapsto A(t)^m$$

satisfying the following conditions:

1. $A(t)^0 = 1$,
2. $A(t)^1 = A(t)$,
3. $(A(t)\cdot B(t))^m = A(t)^m \cdot B(t)^m$,
4. $A(t)^{m+m'} = A(t)^m \cdot A(t)^{m'}$,
5. $A(t)^{mm'} = (A(t)^m)^{m'}$,
6. $(1+t)^m = 1 + mt + O(t^2)$,
7. $A(t)^m\big|_{t\to t^e} = A(t^e)^m$.

Throughout we use the following:

Notation 2.1.8. Partitions $\alpha \vdash n$ are written as $\alpha = (1^{\alpha_1}\cdots i^{\alpha_i}\cdots n^{\alpha_n})$, meaning that there are α_i parts of size i. In particular we recover $n = \sum_i i\alpha_i$. The *automorphism group* of α is the product of symmetric groups $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$. We set $\|\alpha\| = \sum_i \alpha_i$. We denote by $\mathcal{P}$ the set of all partitions (of varying length).

Example 2.1.9. If $R = \mathbb{Z}$, $A(t) = 1 + \sum_{n>0} A_n t^n \in \mathbb{Z}[\![t]\!]$ and $m \in \mathbb{N}$, the known formula [42, p. 40]

$$A(t)^m = \sum_{\alpha\in\mathcal{P}} \left(\prod_{i=0}^{\|\alpha\|-1} (m-i) \cdot \frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!} \right) t^{|\alpha|} = 1 + \sum_{n>0}\sum_{\alpha\vdash n} \left(\prod_{i=0}^{\|\alpha\|-1} (m-i) \cdot \frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!} \right) t^n$$

defines a power structure on $\mathbb{Z}$. For $m \in \mathbb{Z}$, say $m = k - l$ with $k, l \in \mathbb{N}$ it is enough to define $A(t)^m = A(t)^k \cdot (A(t)^l)^{-1}$.

Gusein-Zade, Luengo and Melle-Hernández have proved [24, Thm. 2] that there is a unique power structure

$$(A(t), m) \mapsto A(t)^m$$

on $K_0(\mathsf{Var}_{\mathbb{C}})$ extending the one defined in *loc. cit.* on the semiring $S_0(\mathsf{Var}_{\mathbb{C}})$ of effective motives. The latter is given by the formula

$$A(t)^{[X]} = 1 + \sum_{n>0} \sum_{\alpha \vdash n} \pi_{G_\alpha} \left(\left[\prod_i X^{\alpha_i} \setminus \Delta \right] \cdot \prod_i A_i^{\otimes \alpha_i} \right) t^n. \tag{2.1.10}$$

Here, $\Delta \subset \prod_i X^{\alpha_i}$ is the "big diagonal" (the locus in the product where at least two entries are equal), and the product in big round brackets is a G_α-equivariant motive in $\widetilde{K}_0^{G_\alpha}(\mathsf{Var}_{\mathbb{C}})$, thanks to the power map (2.1.7).

An easy but somewhat crucial remark is that, if $A(t) = (1-t)^{-1}$, one has

$$A(t)^{[X]} = (1-t)^{-[X]} = \zeta_X(t),$$

where ζ_X is the Kapranov zeta function introduced in (2.1.2).

Remark 2.1.10. We will not encounter non-effective coefficients here, so we will have direct access to Formula (2.1.10).

2.1.4 Motivic exponential

The *motivic exponential* is a group isomorphism

$$\mathsf{Exp}\colon t K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!] \xrightarrow{\sim} 1 + t K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!], \tag{2.1.11}$$

converting sums into products and preserving effectiveness. If $A = \sum_{n>0} A_n t^n$ is an effective power series, one has by definition

$$\mathsf{Exp}\left(\sum_{n>0} A_n t^n \right) = \prod_{n>0} \left(1 - t^n\right)^{-A_n},$$

and if A and B are effective, one sets

$$\mathsf{Exp}(A - B) = \prod_{n>0} \left(1-t^n\right)^{-A_n} \cdot \left(\prod_{n>0} \left(1-t^n\right)^{-B_n} \right)^{-1}. \tag{2.1.12}$$

More generally, if $(S, \nu\colon S \times S \to S)$ is a commutative monoid in the category of schemes, with a submonoid $S_+ \subset S$ such that the induced map $\coprod_{n \geqslant 1} S_+^{\times n} \to S$ is of finite type, we similarly define

$$\mathsf{Exp}_\nu(A) = \sum_{n \geqslant 0} \sigma_\nu^n(A)$$

on effective classes, and for A and B two effective classes, we define $\mathsf{Exp}_\nu(A - B)$ by the analogue of (2.1.12), i.e. by $\mathsf{Exp}_\nu(A) \cdot \mathsf{Exp}_\nu(B)^{-1}$. If $(S, \nu) = (\mathbb{N}, +)$, we simply write Exp instead of Exp_+, as we recover (2.1.11).

Example 2.1.11. For any variety X, we have

$$\mathsf{Exp}[X]t = (1-t)^{-[X]} = \zeta_X(t).$$

Caution 2.1.12. Note that Exp does not behave well with respect to variable substitution, e.g. $\mathsf{Exp}(t) = (1-t)^{-1}$ but $\mathsf{Exp}(-t) = 1 - t \neq \mathsf{Exp}(t)|_{t \to -t} = (1+t)^{-1}$.

2.2 The Denef–Loeser motivic zeta function

Denote by J_n and J_∞ the functors of n-jets and the arc space respectively. Let Y be a smooth variety of dimension $d \geqslant 1$, and let $f\colon Y \to \mathbb{A}^1$ be a nonzero regular function with associated hypersurface $Y_0 = f^{-1}(0) \subset Y$. We have an induced morphism $f_n\colon J_n Y \to J_n\mathbb{A}^1 = \mathbb{A}^{n+1}$. Consider the 'order function'

$$J_n\mathbb{A}^1 \xrightarrow{\mathsf{ord}_t} \mathbb{N}\cup\{\infty\}$$

sending an arc $\gamma \in J_n\mathbb{A}^1$ (viewed as an element $\gamma = \sum_{0\leqslant e\leqslant n}\gamma_e t^e \in \mathbb{C}[t]/t^{n+1}$) to

$$\mathsf{ord}_t(\gamma) = \begin{cases} \min\left\{ e \in \mathbb{N} \,\middle|\, \gamma_e \neq 0\right\} & \text{if } \gamma \neq 0 \\ \infty & \text{if } \gamma = 0. \end{cases} \tag{2.2.1}$$

Consider the locally closed subsets

$$\mathcal{X}_{f,n} = \left\{ \gamma \in J_n Y \,\middle|\, \mathsf{ord}_t\, f_n(\gamma) = n \right\} \subset J_n Y$$

for $n \geqslant 1$. These are naturally Y_0-varieties via the projection $\tau_0^n\colon J_n Y \to Y$ sending an arc to its base point, so we have well-defined relative motives $[\mathcal{X}_{f,n} \to Y_0] \in K_0(\mathsf{Var}_{Y_0})$.

Consider, for $n \geqslant 1$, the function

$$\begin{aligned} \mathcal{X}_{f,n} &\xrightarrow{\overline{f}_n} \mathbb{G}_m \\ \gamma &\longmapsto a_n \end{aligned}$$

defined by sending an arc γ such that $\mathsf{ord}_t\, f_n(\gamma) = a_n t^n$ to the scalar $a_n \in \mathbb{G}_m$. Form the fibre

$$\begin{tikzcd} \mathcal{X}_{f,n,1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{f,n} \arrow[d, "\overline{f}_n"] \\ 1 \arrow[r, hook] & \mathbb{G}_m \end{tikzcd}$$

so that, set-theoretically, $\mathcal{X}_{f,n,1}$ consists of arcs γ such that $f_n(\gamma) = t^n$. Now, there are actions

$$\begin{tikzcd} \mathbb{G}_m \times \mathcal{X}_{f,n} \arrow[r, "\cdot"] & \mathcal{X}_{f,n} \\ \mu_n \times \mathcal{X}_{f,n,1} \arrow[r, "\cdot"] \arrow[u, hook] & \mathcal{X}_{f,n,1} \arrow[u, hook] \end{tikzcd}$$

defined as follows. Take $\lambda \in \mathbb{G}_m$ and $\gamma \in \mathcal{X}_{f,n}$, so that $f_n(\gamma) = a t^n$ for some $a \in \mathbb{G}_m$ — in other words, $a = \overline{f}_n(\gamma)$. Define the arc $\lambda\cdot\gamma\colon \operatorname{Spec}\mathbb{C}[t]/t^{n+1} \to Y$ as

$$\operatorname{Spec}\mathbb{C}[t]/t^{n+1} \xrightarrow{\lambda\cdot} \operatorname{Spec}\mathbb{C}[t]/t^{n+1} \xrightarrow{\gamma} Y,$$

where the first isomorphism sends $t \mapsto \lambda t$. This way, we have that $f\circ(\lambda\cdot\gamma)\colon \operatorname{Spec}\mathbb{C}[t]/t^{n+1} \to \mathbb{A}^1 = \operatorname{Spec}\mathbb{C}[t]$ corresponds to

$$\mathbb{C}[t]/t^{n+1} \ni f\circ(\lambda\cdot\gamma)(t) = f\circ\gamma(\lambda t) = a(\lambda t)^n = \lambda^n(at^n) = \lambda^n(f\circ\gamma(t)).$$

In other words,

$$\overline{f}_n(\lambda\cdot\gamma) = \lambda^n a,$$

which confirms at once that $\lambda \cdot \gamma$ has order n and that $\mu_n = \operatorname{Spec} \mathbb{C}[t]/(t^n - 1)$ acts on $\mathcal{X}_{f,n,1}$ by restriction. We obtain equivariant classes

$$[\mathcal{X}_{f,n,1} \longrightarrow Y_0, \hat{\mu}] \in \mathcal{M}_{Y_0}^{\hat{\mu}}$$

where $\hat{\mu} = \varprojlim \mu_n$. Define the *motivic zeta function* attached to the pair (Y, f) as the power series

$$\mathsf{Z}_f(T) = \sum_{n \geqslant 1} [\mathcal{X}_{f,n,1} \to Y_0, \hat{\mu}] \mathbb{L}^{-nd} T^n \in \mathcal{M}_{Y_0}^{\hat{\mu}}.$$

This power series is rational, as we now explain. The key result is the principalisation theorem.

THEOREM 2.2.1 ([4, Thm. 1.10]). *Let Y be a smooth variety, $\mathscr{I} \subset \mathscr{O}_Y$ an ideal sheaf with associated closed subscheme $Z \subset Y$. There exists a proper birational morphism*

$$\widetilde{Y} \xrightarrow{\pi} Y$$

such that

(1) *π is a composition of blowups with smooth centres, and*

(2) *the ideal sheaf $\widetilde{\mathscr{I}} = \pi^* \mathscr{I} \cdot \mathscr{O}_{\widetilde{Y}} \subset \mathscr{O}_{\widetilde{Y}}$ defines a simple normal crossing divisor $\pi^{-1} Z \subset \widetilde{Y}$.*

Consider any embedded resolution $\pi \colon \widetilde{Y} \to Y$ of (Y, f) as in Theorem 2.2.1. Letting Y_0 taking the role of Z, suppose $\pi^{-1} Y_0$ is of the form $\sum_{i \in J} N_i E_i$ when viewed as a simple normal crossing divisor. Also define integers ν_i by the identity $K_\pi = \sum_{i \in J} (\nu_i - 1) E_i$. For a subset $I \subset J$, set

$$m_I = \gcd(N_i \mid i \in I).$$

Form the diagram

$$\begin{tikzcd} \widetilde{Y}_I^{\nu} \arrow[r] & \widetilde{Y} \times_{\mathbb{A}^1_u} \mathbb{A}^1_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Y} \arrow[d, "f \circ \pi"] \\ & \mathbb{A}^1_t \arrow[r] & \mathbb{A}^1_u \end{tikzcd}$$

where $\mathbb{A}^1_t \to \mathbb{A}^1_u$ sends $u \mapsto t^{m_I}$ and $\widetilde{Y}_I^{\nu} \to \widetilde{Y} \times_{\mathbb{A}^1_u} \mathbb{A}^1_t$ is the normalisation map. For $i \in J$ and a nonempty subset $\emptyset \neq I \subset J$, consider the (smooth) subvarieties of $\widetilde{Y}$ defined as

$$E_i^\circ = E_i \setminus \bigcup_{j \neq i} E_j$$
$$E_I = \bigcap_{i \in I} E_i$$
$$E_I^\circ = E_I \setminus \bigcup_{j \in J \setminus I} E_j.$$

The map g_I in the pullback diagram

$$\begin{tikzcd} \widetilde{E}_I^\circ \arrow[r] \arrow[d, "g_I"'] \arrow[dr, phantom, "\square"] & \widetilde{Y}_I^{\nu} \arrow[d] \\ E_I^\circ \arrow[r, hook] & Y \end{tikzcd}$$

is an unramified μ_{m_I}-Galois cover, so that in particular we get a good $\hat{\mu}$-action on $\widetilde{E}_I^\circ$. We thus obtain equivariant classes

$$[\widetilde{E}_I^\circ \longrightarrow Y_0, \hat{\mu}] \in \mathcal{M}_{Y_0}^{\hat{\mu}}$$

THEOREM 2.2.2 ([14, 33]). *In $\mathcal{M}_{Y_0}^{\hat\mu}[\![T]\!]$ there is an identity*

$$\mathsf{Z}_f(T) = \sum_{\emptyset \neq I \subset J} (\mathbb{L}-1)^{|I|-1} [\widetilde{E}_I^\circ \to Y_0, \hat\mu] \prod_{i\in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1-\mathbb{L}^{-\nu_i} T^{N_i}}.$$

For every point $x \in Y_0$, there is a "fibre map"

$$\mathcal{M}_{Y_0}^{\hat\mu} \xrightarrow{\mathrm{Fib}_x} \mathcal{M}_{\mathbb{C}}^{\hat\mu}$$

defined on generators by $[U \to Y_0, \hat\mu] \mapsto [U \times_{Y_0} \kappa(x), \hat\mu]$.

Definition 2.2.3 ([13, §3])**.** Given $f : Y \to \mathbb{A}^1$ as above,

1. $\psi_f = -\lim_{T\to\infty} \mathsf{Z}_f(T) \in \mathcal{M}_{Y_0}^{\hat\mu}$ is called the *motivic nearby fibre*,
2. $\mathsf{MF}_{f,x} = \mathrm{Fib}_x(\psi_f) \in \mathcal{M}_{\mathbb{C}}^{\hat\mu}$ is called the *motivic Milnor fibre* of f at x, and
3. $\phi_f = \psi_f - 1 = \psi_f - [\mathrm{id}\colon Y_0 \to Y_0] \in \mathcal{M}_{Y_0}^{\hat\mu}$ is called the *motivic vanishing cycle* of f.

Note that the limit of $\mathsf{Z}_f(T)$ makes sense because the motivic zeta function is rational by Theorem 2.2.2. More precisely, to obtain ψ_f one has to expand $\mathsf{Z}_f(T)$ in T^{-1} and take minus its constant term. In fact, one has

$$\psi_f = \sum_{\emptyset \neq I \subset J} (1-\mathbb{L})^{|I|-1} [\widetilde{E}_I^\circ \to Y_0, \hat\mu] \in \mathcal{M}_{Y_0}^{\hat\mu}.$$

2.3 Białynicki-Birula decomposition

We recall the classical result of Białynicki-Birula. Recall that a smooth $\mathbb{G}_m$-variety X is *semiprojective* (with respect to the considered action) if the following conditions are satisfied:

(1) the fixed locus $X^{\mathbb{G}_m}$ is proper, and

(2) for every $x \in X$, the limit $\lim_{t\to 0} t \cdot x$ exists in X.

In particular, if X is a projective $\mathbb{G}_m$-variety, then it is automatically semiprojective.

The second condition means the following. Fix $x \in X$. To say that 'the limit exists' means that the family

$$\begin{aligned} \mathbb{G}_m &\xrightarrow{t\cdot} X \\ t &\longmapsto t\cdot x \end{aligned}$$

extends over the origin to a morphism $\mathbb{A}^1 \to X$. For X a moduli space such as the Hilbert scheme, this is clearly a flatness condition: for instance, and this will be the key example for us later, given $[I] \in \mathsf{Hilb}^n(\mathbb{A}^2)$, we say that the limit $\lim_{t\to 0} t\cdot[I]$ exists in $\mathsf{Hilb}^n(\mathbb{A}^2)$ if there is a commutative diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "t\cdot"] \arrow[d, hook] & \mathsf{Hilb}^n(\mathbb{A}^2) \\ \mathbb{A}^1 \arrow[ur, dotted] & \end{tikzcd}$$

Note that the morphism $\mathbb{G}_m \to \mathsf{Hilb}^n(\mathbb{A}^2)$ *is there* by default: it is the restriction of the action to the slice $\mathbb{G}_m \times [I]$. The morphism $\mathbb{A}^1 \to \mathsf{Hilb}^n(\mathbb{A}^2)$ exists if and only if the $\mathbb{G}_m$-family extends *flatly* to $\mathbb{A}^1$. Finally, it is worth noting that such condition depends on the $\mathbb{G}_m$-action: being semiprojective is a property of the $\mathbb{G}$-action on X, not of X alone!

THEOREM 2.3.1 (Białynicki-Birula [3, Sec. 4]). *Let X be a smooth semiprojective $\mathbb{G}_m$-variety. Let*

$$X^{\mathbb{G}_m} = \coprod_{1\leqslant i\leqslant r} X_i$$

be the decomposition of the fixed locus $X^{\mathbb{G}_m} \subset X$ in its connected components. Then,

(1) *for each $i = 1,\ldots,r$ there exists a unique smooth locally closed $\mathbb{G}_m$-invariant subscheme $X_i^+ \subset X$ along with a morphism $\gamma_i : X_i^+ \to X_i$ such that*

- *X_i is a closed subscheme of X_i^+,*
- *γ_i is an affine fibre bundle.*

(2) *for every closed point $x \in X_i$, one has*

$$T_x(X_i^+) = T_x(X)^{\mathsf{fix}} \oplus T_x(X)^+,$$

where $T_x(X)^{\mathsf{fix}}$, resp. $T_x(X)^+$, denotes the $\mathbb{G}_m$-fixed, resp. positive part of $T_x(X)$. In particular, since $T_x(X)^{\mathsf{fix}} = T_xX_i$, the rank of $\gamma_i : X_i^+ \to X_i$ is equal to $\dim_{\mathbb{C}} T_x(X)^+$ for $x \in X_i$.

(3) *The natural morphism*

$$\coprod_{1\leqslant i\leqslant r} X_i^+ \longrightarrow X$$

is bijective. In particular, there is an identity

$$[X] = \sum_{1\leqslant i\leqslant r} [X_i]\mathbb{L}^{\mathrm{d}_i^+} \tag{2.3.1}$$

in $K_0(\mathsf{Var}_{\mathbb{C}})$, where

$$\mathrm{d}_i^+ = \operatorname{rank}\gamma_i = \dim_{\mathbb{C}} T_x(X)^+$$

for an arbitrary closed point $x \in X_i$.

The proof of this theorem is constructive: one defines

$$X_i^+ = \left\{ x \in X \;\middle|\; \lim_{t\to 0} t\cdot x \in X_i \right\} \subset X.$$

Example 2.3.2. Let $X = \mathbb{P}^2 = \operatorname{Proj}\mathbb{C}[x,y,z]$. Consider the action of $\mathbb{G}_m$ with weights $(2,1,0)$. Then $p_1 = (1:0:0)$, $p_2 = (0:1:0)$ and $p_3 = (0:0:1)$ are the fixed points. We have

$$X_i^+ = \left\{ p \in \mathbb{P}^2 \;\middle|\; \lim_{t\to 0} t\cdot p = p_i \right\}$$

for $i = 1,2,3$. Therefore we have 3 cells which are affine spaces, as the fixed locus is isolated. But $\lim_{t\to 0} t\cdot(a:b:c) = \lim_{t\to 0}(t^2a:tb:c)$, so the cells X_i^+ are as follows.

1. $i = 3$: Any point with $c \neq 0$ flows to p_3, as

$$(t^2a:tb:c) \xrightarrow{t\to 0} (0:0:c) = (0:0:1).$$

   Therefore $X_3^+ = \mathrm{D}_+(z) = \mathbb{A}^2$.

2. $i = 1$: the only $p = (a:b:c)$ that flows to $p_1 = (1:0:0)$ is p_1 itself, so $X_1^+ = \mathrm{pt}$.

3. $i = 2$: here we look at points $p = (a:b:c)$ with $c = 0$ (otherwise we automatically flow to p_3, see item above) and $b \neq 0$ (otherwise we flow to p_1), and we want $\lim_{t\to 0}(t^2a:tb:0) = \lim_{t\to 0}(ta:b:0) = (0:1:0)$, so $X_2^+ = \mathrm{V}_+(z)\cap \mathrm{D}_+(y) = \operatorname{Spec}\mathbb{C}[x/y] \cong \mathbb{A}^1$.

All in all, the Białynicki-Birula decomposition of $\mathbb{P}^2$ with the above action[1] is the classical cell decomposition $\mathbb{P}^2 = \mathbb{A}^2 \amalg \mathbb{A}^1 \amalg \mathrm{pt}$.

Example 2.3.3. The smooth variety $X = \mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$ with the action $t\cdot(a,b) = (ta, t^{-1}b)$ is *not* semiprojective. Indeed, for $p = (a,b) \in \mathrm{D}(y) \subset \mathbb{A}^2$, one has

$$\lim_{t\to 0} t\cdot p = \lim_{t\to 0}(ta, t^{-1}b) = (0,\infty) \notin \mathbb{A}^2.$$

One can still construct the (unique) BB cell

$$X_0^+ = \left\{ p \in \mathbb{A}^2 \,\middle|\, \lim_{t\to 0} t\cdot p = 0 \right\}$$

corresponding to the unique fixed point $0 \in \mathbb{A}^2$, but X_0^+ is isomorphic to $\operatorname{Spec}\mathbb{C}[x,y]/y = \mathbb{A}^1$. Therefore the (unique) BB cell does not cover X.

The previous example can be remedied as follows.

Example 2.3.4. The smooth variety $X = \mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$ with the action $t\cdot(a,b) = (t^N a, tb)$ such that $N \gg 0$ *is* semiprojective. Indeed, given the unique fixed point $0 \in \mathbb{A}^2$, we have

$$X_0^+ = \left\{ (a,b) \in \mathbb{A}^2 \,\middle|\, \lim_{t\to 0}(t^N a, tb) = 0 \right\} = \mathbb{A}^2.$$

In other words, every point flows to the origin and no point 'escapes at infinity' as in the previous example.

2.4 The motive of the Quot scheme of points on a curve

PROPOSITION 2.4.1. *Let C be a smooth projective curve over $\mathbb{C}$. Fix a locally free sheaf E of rank r on C. Then $\mathsf{Quot}_C(E,n)$ is a smooth projective variety of dimension rn.*

Proof. Let $E \twoheadrightarrow Q$ be a surjection with kernel K. To confirm smooothness, it is enough to show that $\mathrm{Ext}^1(K,Q) = 0$. But $\mathrm{Ext}^1(K,Q)^\vee = \mathrm{Hom}(Q, K\otimes\omega_C)$, which vanishes since Q is 0-dimensional and $K\otimes\omega_C$ is torsion-free. The dimension is computed as $\dim\mathrm{Hom}(K,Q) = h^0(C, K^\vee\otimes Q) = h^0(C, Q^{\oplus r}) = h^0(C,Q)^{\oplus r} = rn$. □

We wish to apply Theorem 2.3.1 to $\mathsf{Quot}_C(E,n)$. The first step is to endow the Quot scheme with a torus action. For this, we need E to be split.

LEMMA 2.4.2 (Rank reduction). *Let X be an algebraic variety. Let $E = L_1 \oplus\cdots\oplus L_r$ be a split vector bundle over X. Then the algebraic torus $\mathbb{T} = \mathbb{G}_m^r$ acts on $\mathsf{Quot}_X(E,n)$ and*

$$\mathsf{Quot}_X(E,n)^{\mathbb{T}} = \coprod_{n_1+\cdots+n_r = n} \prod_{i=1}^r \mathsf{Hilb}^{n_i}(X).$$

Proof. This is a special case of the main result in [5]. □

Let $X = C$ be a curve from now on, and let $E = L_1\oplus\cdots\oplus L_r$ be a split vector bundle. The connected component of $\mathsf{Quot}_C(E,n)^{\mathbb{T}}$ indexed by the r-tuple $\boldsymbol{n} = (n_1,\ldots,n_r)$ will be denoted $Q_{\boldsymbol{n}}$. Then Theorem 2.3.1 tells us that we have a stratification

$$\mathsf{Quot}_C(E,n) = \coprod_{\boldsymbol{n}} Q_{\boldsymbol{n}}^+$$

[1] You will get the same result if you take arbitrary decreasing weights $\alpha > \beta > 0$.

where $Q_{\boldsymbol{n}}^+ \to Q_{\boldsymbol{n}}$ is an affine bundle. Since

$$Q_{\boldsymbol{n}} \cong \prod_{\alpha=1}^{r} \mathsf{Sym}^{n_\alpha}(C),$$

the motive of the base is known. We only need to compute the dimension of the fibre of $Q_{\boldsymbol{n}}^+ \to Q_{\boldsymbol{n}}$. Let $p \in Q_{\boldsymbol{n}} \subset \mathsf{Quot}_C(E,n)^{\mathbb{T}}$ be a fixed point, so that

$$T_p \mathsf{Quot}_C(E,n) = \mathrm{Hom}_C(K,T)$$

is a $\mathbb{T}$-representation. By Lemma 2.4.2, p has the form

$$p = (K_\alpha \hookrightarrow L_\alpha \twoheadrightarrow T_\alpha)_{\alpha=1,\ldots,r}$$

where of course $T = \bigoplus_\alpha T_\alpha$ and $\chi(T_\alpha) = n_\alpha$. Then

$$\mathrm{Hom}_C(K,T) = \bigoplus_{1\leqslant\alpha,\beta\leqslant r} \mathrm{Hom}_C(K_\alpha, T_\beta).$$

As a $\mathbb{T}$-representation, we have

$$\begin{aligned} T_p \mathsf{Quot}_C(E,n) &= \bigoplus_{1\leqslant\alpha,\beta\leqslant r} \mathrm{Hom}(K_\alpha \otimes w_\alpha, T_\beta \otimes w_\beta) \\ &= \bigoplus_{1\leqslant\alpha,\beta\leqslant r} \mathrm{Hom}_C(K_\alpha, T_\beta) \otimes w_\beta w_\alpha^{-1}. \end{aligned}$$

The positive part of the tangent space, then, namely the part with positive weights, is computed as

$$T_p^+ \mathsf{Quot}_C(E,n) = \bigoplus_{\alpha<\beta} \mathrm{Hom}_C(K_\alpha, T_\beta) \otimes w_\beta w_\alpha^{-1}.$$

We are interested in the dimension of this vector space. We compute

$$\begin{aligned} \dim T_p^+ \mathsf{Quot}_C(E,n) &= \sum_{\alpha<\beta} \dim \mathrm{Hom}_C(K_\alpha, T_\beta) \\ &= \sum_{\alpha<\beta} \dim \mathrm{H}^0(C, K_\alpha^* \otimes_{\mathscr{O}_C} T_\beta) \\ &= \sum_{\alpha<\beta} n_\beta \\ &= \sum_{1\leqslant\beta\leqslant r} (\beta-1) n_\beta. \end{aligned}$$

It follows from Equation (2.3.1) that

$$\begin{aligned} [Q_{\boldsymbol{n}}^+] &= \prod_{\alpha=1}^{r} [\mathsf{Sym}^{n_\alpha}(C)] \cdot \mathbb{L}^{\sum_{1\leqslant\beta\leqslant r}(\beta-1)n_\beta} \\ &= \prod_{\alpha=1}^{r} [\mathsf{Sym}^{n_\alpha}(C)] \cdot \prod_{\alpha=1}^{r} \mathbb{L}^{(\alpha-1)n_\alpha} \\ &= \prod_{\alpha=1}^{r} [\mathsf{Sym}^{n_\alpha}(C)] \mathbb{L}^{(\alpha-1)n_\alpha}. \end{aligned} \tag{2.4.1}$$

Form the generating function

$$\mathsf{Quot}_E(t) = \sum_{n\geqslant 0} [\mathsf{Quot}_C(E,n)] t^n.$$

We can now prove the following.

THEOREM 2.4.3. *There is an identity*

$$\mathsf{Quot}_E(t)=\prod_{\alpha=1}^{r}\zeta_C(\mathbb{L}^{\alpha-1}t).$$

Proof. We compute

$$\begin{aligned}\mathsf{Quot}_E(t)&=\sum_{n\geqslant 0}t^n\sum_{n_1+\cdots+n_r=n}[Q_{\boldsymbol{n}}^+]\\&=\sum_{n\geqslant 0}t^n\sum_{n_1+\cdots+n_r=n}\prod_{\alpha=1}^{r}[\mathsf{Sym}^{n_\alpha}(C)]\mathbb{L}^{(\alpha-1)n_\alpha}\\&=\sum_{n_1,\ldots,n_r\geqslant 0}\prod_{\alpha=1}^{r}[\mathsf{Sym}^{n_\alpha}(C)]\mathbb{L}^{(\alpha-1)n_\alpha}t^{n_\alpha}\\&=\sum_{n_1,\ldots,n_r\geqslant 0}\prod_{\alpha=1}^{r}[\mathsf{Sym}^{n_\alpha}(C)](\mathbb{L}^{\alpha-1}t)^{n_\alpha}\\&=\prod_{\alpha=1}^{r}\zeta_C(\mathbb{L}^{\alpha-1}t).\end{aligned}$$

□

In terms of the motivic exponential, since we have $\mathsf{Exp}([X]t)=(1-t)^{-[X]}=\zeta_X(t)$ for any variety X and we have the transformation rule

$$\mathsf{Exp}(\mathbb{L}^i[X]t)=(1-t)^{-\mathbb{L}^i[X]}=(1-\mathbb{L}^it)^{-[X]}=\zeta_X(\mathbb{L}^it),$$

we may rewrite

$$\mathsf{Quot}_E(t)=\prod_{\alpha=1}^{r}\mathsf{Exp}(\mathbb{L}^{\alpha-1}[C]t)=\mathsf{Exp}\left(\sum_{\alpha=1}^{r}\mathbb{L}^{\alpha-1}[C]t\right)=\mathsf{Exp}\big([\mathbb{P}^{r-1}\times C]t\big).$$

With very little effort, one can show via power structures that this formula holds for all smooth *quasiprojective* curves and for all locally free sheaves E, not necessarily split.

2.5 The motive of the Hilbert scheme of points on $\mathbb{A}^2$

Let S be a smooth complex quasiprojective surface. Consider the generating function

$$\mathsf{Hilb}_S(t)=\sum_{n\geqslant 0}[\mathsf{Hilb}^n(S)]t^n\in K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!],$$

In this section we shall find a formula for $\mathsf{Hilb}_S(t)$ in the case $S=\mathbb{A}^2$. The case of an arbitrary surface will be handled in **??** thanks to the power structure machinery.

We know that $\mathsf{Hilb}^n(\mathbb{A}^2)$ is acted on by $\mathbb{T}=\mathbb{G}_m^2$ with isolated fixed locus. Each fixed point corresponds to a partition of n, i.e. a Young diagram λ of size $|\lambda|=n$. Explicitly,

$$I_\lambda=\left\langle x^iy^j\,\middle|\,(i,j)\notin\lambda\right\rangle\subset\mathbb{C}[x,y]$$

is the ideal corresponding to $\lambda\vdash n$. The first step is to use a subtorus $\mathbb{G}_m\subset\mathbb{T}$ with the same fixed locus. This can be achieved by acting with *algebraic* (on functions) weights $(N,1)$ such that $N\gg 1$, namely setting up the action

$$t\cdot(x,y)=(t^Nx,ty),$$

so that $t\in\mathbb{G}_m$ acts on an ideal $[I]\in\mathsf{Hilb}^n(\mathbb{A}^2)$ via

$$t\cdot I=\left\{f(t^Nx,ty)\,\middle|\,f(x,y)\in I\right\}\subset\mathbb{C}[x,y]. \tag{2.5.1}$$

We already know that the action with weights $(1,-1)$ does not make $\mathbb{A}^2 = \mathsf{Hilb}^1(\mathbb{A}^2)$ semiprojective, cf. Example 2.3.4, and one can show the same for $n > 1$ as well. However, we have the following result.

LEMMA 2.5.1. *The smooth scheme* $\mathsf{Hilb}^n(\mathbb{A}^2)$ *along with the* $\mathbb{G}_m$*-action* (2.5.1) *is semiprojective for all* $n \geqslant 1$.

Proof. Clearly there is an identity $\mathsf{Hilb}^n(\mathbb{A}^2)^{\mathbb{T}} = \mathsf{Hilb}^n(\mathbb{A}^2)^{\mathbb{G}_m}$, so properness of the fixed locus is immediate. We need to confirm that limits exist. Let $[I] \in \mathsf{Hilb}^n(\mathbb{A}^2)$ be a point. We have to extend the map

$$\begin{aligned} \mathbb{G}_m &\longrightarrow \mathsf{Hilb}^n(\mathbb{A}^2) \\ t &\longmapsto t\cdot[I] \end{aligned}$$

to $\mathbb{A}^1$.

For a monomial $x^a y^b$, set

$$\mathsf{wt}(x^a y^b) = aN + b.$$

Let $f = \sum_{a,b} c_{ab} x^a y^b \in R = \mathbb{C}[x,y]$. Define

$$\mathsf{m}_f = \min\left\{ \mathsf{wt}(x^a y^b) \,\middle|\, c_{ab} \neq 0 \right\},$$

so that

$$\begin{aligned} t\cdot f &= \sum_{a,b} c_{ab} t^{aN+b} x^a y^b = \sum_{a,b} c_{ab} t^{\mathsf{wt}(x^a y^b)} x^a y^b \\ &= t^{\mathsf{m}_f}\left(\sum_{\mathsf{wt}(x^a y^b)=\mathsf{m}_f} c_{ab} x^a y^b + \text{higher weight terms} \right). \end{aligned}$$

Define

$$\mathsf{in}_{(N,1)}(f) = \sum_{\mathsf{wt}(x^a y^b)=\mathsf{m}_f} c_{ab} x^a y^b$$

to be the *initial form* of f with respect to the weights $(N,1)$. Let now $[I] \in \mathsf{Hilb}^n(\mathbb{A}^2)$ be a closed point. Define

$$\mathsf{in}_{(N,1)}(I) = \left\langle \mathsf{in}_{(N,1)}(f) \,\middle|\, f \in I \right\rangle \subset R.$$

This is the *initial ideal* of I with respect to the weights $(N,1)$. It is this ideal that will fill in the $\mathbb{G}_m$-family we started with at the origin.

We wish to construct an ideal $\widetilde{I} \subset R[t] = R \otimes_{\mathbb{C}} \mathbb{C}[t]$ (this corresponds to a closed subscheme $\widetilde{Z} \subset \mathbb{A}^2 \times \operatorname{Spec} \mathbb{C}[t]$) such that

(1) over $t \neq 0$, the ideal $\widetilde{I}_t$ corresponds to the ideal $t \cdot I$, and

(2) $R[t]/\widetilde{I}$ is $\mathbb{C}[t]$-flat.

Consider the (exhaustive, bounded below) filtration on R given by

$$R_{\leqslant d} = \mathsf{Span}\left\{ x^a y^b \,\middle|\, aN + b \leqslant d \right\} \subset R.$$

It satisfies $R_{\leqslant 0} = \mathbb{C}$ and $R_{\leqslant d} \cdot R_{\leqslant e} \subset R_{\leqslant d+e}$. We obtain an induced filtration

$$I_{\leqslant d} = I \cap R_{\leqslant d},$$

which we use to define the *Rees ideal*

$$\widetilde{I} = \bigoplus_{d\geqslant 0} I_{\leqslant d}\, t^d \subset R[t].$$

We leave it to the reader to verify that this ideal has the desired properties. To check flatness, it is in fact enough – after having confirmed (1) – that the colength over the origin is equal to n. $\square$

We are now ready to apply Theorem 2.3.1. Fix $n\geqslant 2$. Define the attracting cells

$$H_\lambda^+ = \left\{ I \in \mathsf{Hilb}^n(\mathbb{A}^2) \,\middle|\, \lim_{t\to 0} t\cdot I = I_\lambda \right\} \subset \mathsf{Hilb}^n(\mathbb{A}^2).$$

Combining Theorem 2.3.1 and Lemma 2.5.1, we obtain a locally closed stratification

$$\mathsf{Hilb}^n(\mathbb{A}^2) = \coprod_{\lambda\vdash n} H_\lambda^+$$

where each stratum H_λ^+ is an affine space of dimension

$$\mathrm{d}_\lambda^+ = \dim_{[I_\lambda]} T^+\,\mathsf{Hilb}^n(\mathbb{A}^2).$$

In particular, we have an identity of motives

$$[\mathsf{Hilb}^n(\mathbb{A}^2)] = \sum_{\lambda\vdash n} \mathbb{L}^{\mathrm{d}_\lambda^+} \in K_0(\mathsf{Var}_{\mathbb{C}}) \tag{2.5.2}$$

for each $n\geqslant 2$.

PROPOSITION 2.5.2. *For a partition $\lambda\in\mathcal{P}$ of size at least 2, there is an identity*

$$\mathrm{d}_\lambda^+ = |\lambda| + \lambda_1,$$

where λ_1 is the number of columns in the Young diagram attached to λ (cf. Notation 1.2.21).

Proof. Let λ be a partition of n, and let $I_\lambda\subset\mathbb{C}[x,y]$ be the corresponding monomial ideal. Denote by $a(\square)$ and $l(\square)$ the arm-length and the leg-length of a box $\square\in\lambda$. The tangent space

$$T_\lambda\,\mathsf{Hilb}^n(\mathbb{A}^2)$$

is a $\mathbb{G}_m^2$-representation, and by Exercise 1.2.33 it admits the expression

$$T_\lambda\,\mathsf{Hilb}^n(\mathbb{A}^2) = \sum_{\square\in\lambda} t_1^{-l(\square)} t_2^{a(\square)+1} + t_1^{l(\square)+1} t_2^{-a(\square)}.$$

Now we restrict to the subtorus

$$\mathbb{G}_m = \left\{ (t^N, t) \,\middle|\, t\in\mathbb{C}^\times \right\} \subset \mathbb{G}_m^2$$

and we count positive tangents. Make the formal substitution $t_1 \mapsto t^N$ and $t_2\mapsto t$. This way, we see that the first bit of the sum gives a contribution whenever $l(\square)=0$. But the boxes with trivial leg-length are as many as the columns of the Young diagram corresponding to λ, so this bit contributes precisely λ_1. On the other hand, in the second bit of the sum every box contributes because t_1 approaches 0 much faster than t_2 and so no restriction on $a(\square)$ is present. This proves the desired formula. $\square$

Note that, if λ is the unique partition of size 1, then $|\lambda|+\lambda_1=2$, therefore Equation (2.5.2) also works in the case $n=1$ if one *defines* d_λ^+ by the relation in Proposition 2.5.2. The *actual* definition of d_λ^+ would give 1! Keeping this in mind, we can now finish the calculation.

COROLLARY 2.5.3. *There are identities*

$$\mathsf{Hilb}_{\mathbb{A}^2}(t) = \prod_{m\geqslant 1} \frac{1}{1-\mathbb{L}^{m+1}t^m} = \mathsf{Exp}\left(\frac{\mathbb{L}^2 t}{1-\mathbb{L}t}\right). \tag{2.5.3}$$

Proof. We have

$$\mathsf{Hilb}_{\mathbb{A}^2}(t) = \sum_{n\geqslant 0} [\mathsf{Hilb}^n(\mathbb{A}^2)]t^n = \sum_{\lambda\in\mathcal{P}} t^{|\lambda|}\mathbb{L}^{|\lambda|+\lambda_1}$$

so the first identity reduces to the next lemma, taken with $a = b = 1$. The second identity is obvious, by the very definition of Exp. □

LEMMA 2.5.4. *For every pair* $(a,b)\in\mathbb{N}\times\mathbb{N}$*, there is an identity*

$$\prod_{m\geqslant 1} \frac{1}{1-\mathbb{L}^{am+b}t^m} = \sum_{\lambda\in\mathcal{P}} t^{|\lambda|}\mathbb{L}^{a|\lambda|+b\lambda_1} \tag{2.5.4}$$

in the subring $A = \mathbb{Z}[\mathbb{L}][\![t]\!] \subset K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!]$.

Proof. Write $\mathsf{F}_m(t) = (1-\mathbb{L}^{am+b}t^m)^{-1}$, so that

$$\mathsf{F}(t) = \prod_{m\geqslant 1} \frac{1}{1-\mathbb{L}^{am+b}t^m} = \prod_{m\geqslant 1} \mathsf{F}_m(t).$$

Note that

$$\mathsf{F}_m(t) = \sum_{k\geqslant 0} \mathbb{L}^{k(am+b)}t^{mk} \in 1 + t^m A,$$

so that $\mathsf{Coeff}_{t^N}\,\mathsf{F}(t) = \mathsf{Coeff}_{t^N}\,\mathsf{F}_1(t)\mathsf{F}_2(t)\cdots\mathsf{F}_N(t)$ for every $N\geqslant 1$. We have

$$\begin{aligned}
\mathsf{Coeff}_{t^N}\,\mathsf{F}_1(t)\mathsf{F}_2(t)\cdots\mathsf{F}_N(t) &= \mathsf{Coeff}_{t^N} \prod_{1\leqslant m\leqslant N}\sum_{k_m\geqslant 0} \mathbb{L}^{k_m(am+b)}t^{mk_m} \\
&= \sum_{\substack{k_1,\dots,k_N\geqslant 0 \\ \sum_m mk_m = N}} \prod_{1\leqslant m\leqslant N} \mathbb{L}^{k_m(am+b)} \\
&= \sum_{\lambda\vdash N} \mathbb{L}^{\sum_m k_m(am+b)},
\end{aligned}$$

where the last identity identifies the tuple $k_\bullet$ with the partition $\lambda = (1^{k_1}\cdots m^{k_m}\cdots N^{k_N})\vdash N$. We may rewrite

$$\sum_m k_m(am+b) = a\sum_m mk_m + b\sum_m k_m = a|\lambda| + b||\lambda||,$$

and hence

$$\mathsf{F}(t) = \sum_{\lambda\in\mathcal{P}} t^{|\lambda|}\mathbb{L}^{a|\lambda|+b||\lambda||}.$$

But we are summing over all partitions, and on this set there is the involution sending a partition to its transpose λ^t (exchanging rows and columns, and preserving the size). Therefore the formula is equivalent to the sought after one. □

Example 2.5.5. Let us do a sanity check explicit example with $n = 3$. Expanding (2.5.3) to order 4 we find

$$\mathsf{Hilb}_{\mathbb{A}^2}(t) = 1 + \mathbb{L}^2 + \mathbb{L}^3(1+\mathbb{L})t^2 + \mathbb{L}^4(1+\mathbb{L}+\mathbb{L}^2)t^3 + \cdots$$

so we should try and confirm that

$$[\mathsf{Hilb}^3(\mathbb{A}^2)] = \mathbb{L}^4(1+\mathbb{L}+\mathbb{L}^2) = \mathbb{L}^3(\mathbb{L}+\mathbb{L}^2+\mathbb{L}^3)$$

using the tangent representation. There are 3 partitions to consider, namely

$$\ydiagram{3} \qquad \ydiagram{1,1,1} \qquad \ydiagram{2,1}$$

Let us compute $T_\lambda = T_\lambda \operatorname{Hilb}^3(\mathbb{A}^2)$ for all 3 partitions λ. For instance, the first partition has

$$\begin{aligned} a(0,0) &= 2, & l(0,0) &= 0 \\ a(1,0) &= 1, & l(1,0) &= 0 \\ a(2,0) &= 0, & l(2,0) &= 0. \end{aligned}$$

Recall from Equation (2.5.5) that we computed

$$\begin{aligned} T_{\ydiagram{3}} &= t_1 + t_2 + t_2^2 + t_2^3 + t_1 t_2^{-1} + t_1 t_2^{-2} \\ T_{\ydiagram{1,1,1}} &= t_1 + t_2 + t_1^2 + t_1^3 + t_1^{-1} t_2 + t_1^{-2} t_2 \\ T_{\ydiagram{2,1}} &= 2t_1 + 2t_2 + t_1^2 t_2^{-1} + t_1^{-1} t_2^2. \end{aligned} \tag{2.5.5}$$

Sending $t_1 \mapsto t^N$ and $t_2 \mapsto t$, we find

$$\begin{aligned} T_{\ydiagram{3}} &= (t^3 + t^{N-2}) + (t^2 + t^{N-1}) + (t + t^N) & &\to & &\mathbb{L}^6 \\ T_{\ydiagram{1,1,1}} &= (t^{1-2N} + t^{3N}) + (t^{1-N} + t^{2N}) + (t + t^N) & &\to & &\mathbb{L}^4 \\ T_{\ydiagram{2,1}} &= (t^{2-N} + t^{2N-1}) + (t + t^N) + (t + t^N) & &\to & &\mathbb{L}^5 \end{aligned}$$

which confirms the formula (the negative tangents are highlighted in red).

Bibliography

[1] Sjoerd Beentjes and Andrea T. Ricolfi, *Virtual counts on Quot schemes and the higher rank local DT/PT correspondence*, Math. Res. Lett. **28** (2021), no. 4, 967–1032. 1.2.7

[2] Kai Behrend, Jim Bryan, and Balázs Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, Invent. Math. **192** (2013), no. 1, 111–160. 2.1.2

[3] Andrzej Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2) **98** (1973), no. 3, 480–497. 2.3.1

[4] Edward Bierstone and Pierre D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. 2.2.1

[5] Emili Bifet, *Sur les points fixes du schéma* $\mathrm{Quot}_{\mathscr{O}_X^r/X/k}$ *sous l'action du tore* $\mathbf{G}_{m,k}^r$, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), no. 9, 609–612. 2.4

[6] Franziska Bittner, *On motivic zeta functions and the motivic nearby fiber*, Math. Z. **249** (2005), no. 1, 63–83. 2.1.2

[7] Joel Briançon, *Description de* $\mathrm{Hilb}^n\mathbb{C}\{x,y\}$, Invent. Math. **41** (1977), 45–89. 5

[8] Dustin Cartwright, Daniel Erman, Mauricio Velasco, and Bianca Viray, *Hilbert schemes of 8 points*, Algebra Number Theory **3** (2009), no. 7, 763–795. 1.2.3

[9] Alberto Cazzaniga and Andrea T. Ricolfi, *Framed sheaves on projective space and Quot schemes*, Math. Z. **300** (2022), 745–760. 1.2.9

[10] Ben Davison and Andrea T. Ricolfi, *The local motivic DT/PT correspondence*, J. Lond. Math. Soc. **104** (2021), no. 3, 1384–1432. 2.1.1, 2.1.2, 2.1.2

[11] Pierre Deligne, *Théorie de Hodge. II.*, Publ. Math., Inst. Hautes Étud. Sci. **40** (1971), 5–57. 2.1.1

[12] ———, *Théorie de Hodge. III.*, Publ. Math., Inst. Hautes Étud. Sci. **44** (1974), 5–77. 2.1.1

[13] Jan Denef and François Loeser, *Geometry on arc spaces of algebraic varieties*, 3rd European congress of mathematics (ECM), Barcelona, Spain, July 10–14, 2000. Volume I, Basel: Birkhäuser, 2001, pp. 327–348. 2.1.1, 2.2.3

[14] Jan Denef and François Loeser, *Lefschetz numbers of iterates of the monodromy and truncated arcs*, Topology **41** (2002), no. 5, 1031–1040. 2.2.2

[15] Theodosios Douvropoulos, Joachim Jelisiejew, Bernt Ivar Utstøl Nødland, and Zach Teitler, *The Hilbert scheme of 11 points in $\mathbb{A}^3$ is irreducible*, Combinatorial algebraic geometry, Fields Inst. Commun., vol. 80, Fields Inst. Res. Math. Sci., Toronto, ON, 2017, pp. 321–352. 3

[16] David Eisenbud and Joe Harris, *3264 and all that – a second course in Algebraic Geometry*, Cambridge University Press, Cambridge, 2016. 1.1.2

[17] Geir Ellingsrud and Manfred Lehn, *Irreducibility of the punctual quotient scheme of a surface*, Ark. Mat. **37** (1999), no. 2, 245–254. 9

[18] Leonhard Euler, *Introductio in analysin infinitorum*, V. 1, Lausanne, 1748. 1.2.24

[19] Barbara Fantechi and Andrea T. Ricolfi, *On the stack of 0-dimensional coherent sheaves: motivic aspects*, Bull. Lond. Math. Soc. **57** (2025), no. 6, 1607–1649. 1.2.1

[20] ______, *On the stack of 0-dimensional coherent sheaves: structural aspects*, In: Moduli, Motives and Bundles: New Trends in Algebraic Geometry. London Mathematical Society Lecture Note Series **499**, Cambridge University Press., 2025, pp. 1–32. 1.2.1

[21] Nadir Fasola, Sergej Monavari, and Andrea T. Ricolfi, *Higher rank K-theoretic Donaldson–Thomas theory of points*, Forum Math. Sigma **9** (2021), no. E15, 1–51. 1.2.13

[22] William Fulton, *Intersection theory*, Springer, 1984. 0.1

[23] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. 2.1.1

[24] Sabir M. Gusein-Zade, Ignacio Luengo, and Alejandro Melle-Hernández, *A power structure over the Grothendieck ring of varieties*, Math. Res. Lett. **11** (2004), no. 1, 49–57. 2.1.3, 2.1.7, 2.1.3

[25] ______, *Power structure over the Grothendieck ring of varieties and generating series of Hilbert schemes of points*, Mich. Math. J. **54** (2006), no. 2, 353–359. 2.1.3

[26] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Publ. Math. Inst. Hautes Études Sci. **29** (1966), no. 1, 7–48. 1.2.5

[27] ______, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. 0.1

[28] Abdelmoubine A. Henni and Marcos Jardim, *Commuting matrices and the Hilbert scheme of points on affine spaces*, Adv. Geom. **18** (2018), no. 4, 467–482. 3, 1.2.9

[29] Anthony Iarrobino, *Reducibility of the families of* 0*-dimensional schemes on a variety*, Invent. Math. **15** (1972), 72–77. 3

[30] Joachim Jelisiejew and Hanieh Keneshlou, *On construction of k-regular maps to Grassmannians via algebras of socle dimension two*, ArXiv:2112.14106, 2021. 6

[31] Joachim Jelisiejew and Klemen Šivic, *Components and singularities of Quot schemes and varieties of commuting matrices*, J. Reine Angew. Math. **788** (2022), 129–187. 10

[32] Mikhail Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, ArXiv:0001005, 2000. 2.1.1

[33] Eduard Looijenga, *Motivic measures*, Séminaire Bourbaki. Volume 1999/2000. Exposés 865–879, Paris: Société Mathématique de France, 2002, pp. 267–297, ex. 2.1.1, 2.2.2

[34] Percy A. MacMahon, *Combinatory Anaysis*, vols. 1-2, Cambridge University Press, 1915, 1916. Reprinted by Chelsea, New York, 1960. 1.2.26

[35] Guerino Mazzola, *Generic finite schemes and Hochschild cocycles*, Comment. Math. Helv. **55** (1980), no. 2, 267–293. 2

[36] Anton Mellit, *Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures)*, Invent. Math. **221** (2020), no. 1, 301–327. 1

[37] Sergej Monavari and Andrea T. Ricolfi, *On the motive of the nested Quot scheme of points on a curve*, J. Algebra **610** (2022), 99–118. 7

[38] Hiraku Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999. 1.2.9

[39] Andrea T. Ricolfi, *The equivariant Atiyah class*, C. R. Math. Acad. Sci. Paris **359** (2021), no. 3, 257–282. 1.2.13

[40] David Rydh, *Families of cycles and the Chow scheme*, Ph.D. thesis, KTH, Stockholm, 2008. 1.2.1

[41] Stacks Project Authors, *Stacks Project*, 2024. 0.1.6, 1.2.3

[42] Richard P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Univ. Press, Cambridge, 1999. 2.1.9

[43] Michał Szachniewicz, *Non-reducedness of the Hilbert schemes of few points*, ArXiv:2109.11805, 2021. 4

[44] Balázs Szendrői, *Non-commutative Donaldson–Thomas invariants and the conifold*, Geom. Topol. **12** (2008), no. 2, 1171–1202. 1.2.7

[45] Klemen Šivic, *On varieties of commuting triples III*, Linear Algebra Appl. **437** (2012), no. 2, 393–460. 3