HILBERT & QUOT SCHEMES

SISSA ALGEBRAIC GEOMETRY SEMINAR — SPRING 2024

1. The topic, and the goal of the seminar

This semester we focus on Hilbert schemes and Quot schemes.

After reviewing the Quot construction from first principles, we shall see examples: Hilbert schemes of curves in $\mathbb{P}^3$ first, then the case of nested Hilbert schemes, which is particularly explicit thanks to a concrete description in terms of quiver representations. Then we move towards an application of the deformation theory machinery we learned the last semester, allowing us to prove that under certain assumptions $\mathrm{Hilb}^n(X)$ is isomorphic to the moduli space of torsion free sheaves of rank 1 with trivial determinant: this will give a modern proof of the smoothness of $\mathrm{Hilb}^n(S)$, when S is a smooth surface. Once we (re)know this, we study $\mathrm{Hilb}^n(S)$ more in depth, linking its cohomology with representations of the Heisenberg algebra. Then we shall see an important technique in enumerative geometry, namely *degeneration* (25 March). Finally we move to Hilbert schemes on 3-folds, which is when pathologies start to arise. We deal with reducibility on April 8th. Still, the 3-fold case will be rescued by the existence of a POT on the Hilbert scheme of points, which will open the way to the definition of (degree 0) DT invariants of a 3-fold (April 22).

2. Schedule of the seminar

We plan on having 8 talks by the participants. After that, we shall regroup and decide what will come next. Here is the detailed program:

- **12 feb**: Construction of Quot schemes [1]
- **19 feb**: Examples of Hilbert schemes (in particular twisted cubics in $\mathbb{P}^3$) [6, 4]
- **26 feb**: Interlude — Moschetti (guest)
- **4 mar**: Nested Hilbert schemes via quivers
- **11 mar**: Comparison between $\mathrm{Hilb}^n(X)$ and moduli of ideal sheaves. Smoothness of $\mathrm{Hilb}^n(S)$
- **18 mar**: Cohomology of $\mathrm{Hilb}^n(S)$ and Nakajima operators [3, 1]
- **25 mar**: Degeneration techniques for Quot schemes
- **8 apr**: Reducibility of $\mathrm{Hilb}^n(\mathbb{A}^3)$ for $n \geq 78$ [2]
- **15 apr**: Motivic invariants: the motive of the Quot scheme of points [5]
- **22 apr**: Perfect obstruction theory on $\mathrm{Hilb}^n(X)$ for X a smooth 3-fold and DT invariants [6]

References

1. Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained.
2. Anthony Iarrobino, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 337–378.
3. Hiraku Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, vol. 18, American Mathematical Society, Providence, RI, 1999.
4. Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774.
5. Andrea T. Ricolfi, *On the motive of the Quot scheme of finite quotients of a locally free sheaf*, J. Math. Pures Appl. **144** (2020), 50–68.
6. Andrea T. Ricolfi, *An invitation to modern enumerative geometry*, SISSA Springer Ser., vol. 3, Cham: Springer, 2022 (English).