

AG SEMINAR ON MOTIVIC INTEGRATION

10 TALKS OF 60–75 MINUTES

- Arc spaces — 26/10** Let X be a variety. We need to define the **domain of integration**: this is the arc space $J_\infty(X)$. Construct the jet scheme $J_n(X)$ (aka space of n -arcs) affine-locally via $A \mapsto \text{Hom}(\text{Spec } A[t]/t^{n+1}, X)$, explain the gluing (Greenberg’s theorem, check [1] for more references) and then construct the arc space $J_\infty(X) = \varprojlim J_n(X)$ with natural maps $\pi_n: J_\infty(X) \rightarrow J_n(X)$. **Prove local triviality** of truncation morphisms $\tau_n^m: J_m(X) \rightarrow J_n(X)$, $m \geq n$ (assume X smooth). Work out basic examples, e.g. the first jets of the cusp $y^2 = x^3$. Refs: [1, 6, 9]

- Grothendieck rings, cylinders, and motivic measure — 9/11** Introduce $K_0(\text{Var}_\mathbb{C})$, the Lefschetz class $\mathbb{L}$, localisation $K_0(\text{Var}_\mathbb{C})[\mathbb{L}^{-1}]$, and its dimensional completion $\widehat{\mathcal{M}}$, which is the **value ring for motivic integrals**. Mention the non-archimedean valuation perspective (it’s convenient for convergence issues). Define **cylinders** $C \subset J_\infty(X)$ as those sets of the form $\pi_n^{-1}(C_n)$ for $C_n \subset J_n(X)$ constructible. Define their **motivic measure** as $\mu_X(C) = [C_n]\mathbb{L}^{-(n+1)\dim X}$ in $\widehat{\mathcal{M}}$ and define the motivic measure of $J_\infty(X)$. Compute the measure of contact-order strata in an affine space. Refs: [1, 6, 9]

- Measurable sets, functions, and motivic integration — 16/11** Extend motivic measure beyond cylinders, **define measurable sets, functions and integrals**, and discuss convergence. Define

$$\int_A \mathbb{L}^{-f} d\mu_X, \quad A \subset J_\infty(X)$$

for a measurable function f . Measurable set = approximated by cylinders. Prove that the **order function of an ideal sheaf is measurable**. Calculate integrals for monomial functions $x^a y^b$ on $\mathbb{A}^2$ and simple normal crossings divisors (this will be needed later). Refs: [1, 6, 9]

- The birational transformation rule and Kontsevich’s theorem — 23/11** Define the Jacobian ideal of a birational morphism between smooth varieties. Prove the **change-of-variables formula**, do a blow-up example, say on $Bl_0(\mathbb{A}^2) \rightarrow \mathbb{A}^2$. Deduce Kontsevich’s theorem stating that

birational smooth CY manifolds have equal Hodge numbers!

Refs: [1, 6, 9]

- Naive and monodromic motivic zeta functions — 30/11** Define the **zeta function** of a regular function $f: X \rightarrow \mathbb{A}^1$ using the orders of its truncated arcs. Define the **roots of unity action** on truncated arcs, and introduce equivariant Grothendieck rings; calculate the example $f(x) = x^r$. Define more generally the **zeta function of an ideal sheaf** and interpret it via motivic integrals. [1, 6, 9]

- Rationality of zeta functions — 14/12** Use an embedded resolution and the birational transformation rule to obtain the Denef–Loeser formula for the naive motivic zeta function of a regular function, or more generally of an ideal sheaf: this confirms **rationality of the zeta function**. Refs: [5, 11]

- Worked out example: the cusp $f(x, y) = y^2 - x^3$ — 21/12** Compute an embedded resolution of the cusp explicitly. **Compute its local naive motivic zeta function** from the resolution formula, and deduce the associated local topological zeta function. Talk about the poles. Another example if time permits. Ref: [4]

- Motivic nearby fibres, vanishing cycles, and monodromy — 18/1** Define the motivic nearby fibre and **motivic vanishing cycles** as the formal limit of the monodromic zeta function.

Describe its relationship with classical Milnor fibre and monodromy, and revisit the cusp. Refs: [6, 9]

- 9. Motivic Thom–Sebastiani — 25/1** Introduction to **Thom–Sebastiani for vanishing cycles**. Introduce the **convolution product** on monodromic Grothendieck rings using the mapping torus and state the motivic Thom–Sebastiani theorem for sums $f \oplus g$. Outline the proof. Ref: [9]
- 10. A glimpse on the Monodromy Conjecture — 1/2** **State the monodromy conjecture** (topological — see [8] and [6, Conj. 3.4.1] — and motivic [2, Conj. 10.2]) and say something about what’s known about it: 2-dimensional case [3] abelian varieties, K3 surfaces [7], Hilbert schemes of points on surfaces [10].

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