LOCAL DONALDSON–THOMAS INVARIANTS AND THEIR REFINEMENTS

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ABSTRACT

In this thesis we provide some new computations in enumerative and motivic Donaldson–Thomas theory. On the (classical) enumerative side, we compute the zero-dimensional DT theory of abelian threefolds via their Kummer schemes, and the local DT invariants attached to a smooth curve embedded in a projective Calabi–Yau threefold. For the latter, we combine a weighted Euler characteristic calculation for certain Quot schemes with a local study of the Hilbert–Chow morphism. The result is a wall-crossing type formula relating local Donaldson–Thomas invariants to local Pandharipande–Thomas invariants.

On the motivic side, we define motivic DT invariants refining some of the numbers computed earlier. We conjecture a simple motivic DT/PT correspondence refining the enumerative wall-crossing formula obtained previously, and we provide some evidence.

A common approach for both enumerative and motivic calculations is the study of a line in affine three-space: this *local model* carries enough information to study the geometry of an arbitrary smooth curve embedded in a smooth quasi-projective threefold.

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INTRODUCTION

The enumerative geometry of algebraic curves is one of the richest subjects in modern Algebraic Geometry; it is particularly interesting in the case of *curves on threefolds*. Here a huge influence has come and is still coming from Physics, especially String Theory. Heuristically, curves on Calabi–Yau threefolds are expected to move in 0-dimensional families, so one can ask for a suitable technology to count them.

> A *Calabi–Yau threefold* is a smooth quasi-projective complex algebraic variety Y of dimension 3, with a trivialization $\omega_Y \cong \mathscr{O}_Y$.

There are several ways to compactify the space of smooth embedded curves on a threefold, in such a way that the resulting moduli space admits a virtual fundamental class. The existence of such a class is a nontrivial portion of the "technology" mentioned above, needed to define a functioning enumerative theory. See for instance [64] for a survey on this subject, touching upon the conjectures relating the existing curve counting theories.

We will only mention two such theories in this work, namely Donaldson–Thomas theory and Pandharipande–Thomas theory. The former extracts enumerative invariants from the Hilbert scheme, viewed as a moduli space of ideal sheaves, the latter from the moduli space of stable pairs.

Donaldson–Thomas (DT) theory, defined for complex threefolds, was born when Thomas [78] constructed a symmetric perfect obstruction theory on compact moduli spaces of stable sheaves on a threefold with trivial (or negative) canonical class. Thomas also proved deformation invariance of the induced virtual fundamental class. The interesting case for the enumeration of algebraic curves is the ideal sheaf case.

> An *ideal sheaf* is a torsion-free sheaf of rank 1 with trivial determinant.

Pandharipande–Thomas (PT) theory is younger [62], and the moduli space is "smaller" than the Hilbert scheme: no free-roaming points are allowed. Both DT and PT theory are *sheaf* theories. The associated moduli spaces can both be interpreted as moduli spaces of stable objects in the derived category of the ambient threefold. The numerical invariants remain unchanged under small deformations of the complex structure on the underlying threefold, but they are sensitive to a change of stability condition. The rules that govern these changes are the so called wall-crossing formulas [41, 44].

A conjectural equivalence between DT and PT theory was first formulated in [62]. This is the "DT/PT correspondence", proved by Bridgeland [15] and Toda [79]. Bayer interpreted it as a wall-crossing type formula in the sense of polynomial stability [4]. We will explore a version of this correspondence later in this work.

So far we only talked about the numerical aspect of curve counting. But for the sheaf counting theories on Calabi–Yau threefolds, there is a more "refined" aspect, again with roots in theoretical Physics, see for instance [39]. Mathematically, the existence of a refined theory remembering more than just the numbers is suggested by a well precise fact: the obstruction theory used to define the numerical invariants is *symmetric*; in this situation, a theorem of Behrend implies that the associated counting invariants are computable by means of cut and paste techniques, which is a first indication that the numbers may be nothing but a realization of some cohomology theory on the moduli space. This intuition is sustained by the technical result stating that the moduli space is locally a critical locus, that is, locally of the form

$$\left\{\mathrm{d}f = 0\right\} \subset V,$$

for some holomorphic function f on a complex manifold V. The natural symmetric obstruction theory on a critical locus admits a canonical motivic refinement due to Denef and Loeser. None of this holds in Gromov–Witten theory: the obstruction theory on the moduli space of stable maps is not symmetric.

In this thesis, these finer invariants will live in the *ring of motivic weights* $\mathcal{M}_{\mathbb{C}}$, a convenient enhancement of the more familiar Grothendieck ring of varieties. Therefore they will be called *motivic* throughout. The specialization giving us back the numerical DT invariants is simply the Euler characteristic

$$\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}.$$

This thesis deals with the calculation of some *local* DT invariants, and with the construction of natural motivic refinements of these numbers. The word "local" refers to the fact that we *fix* a curve C inside our threefold Y and we study the contribution of that curve to the *global* invariants, which enumerate curves in the whole homology class of C. Our guiding strategy, for both enumerative and motivic calculations, is to exploit the *local model* of a line

$$\mathbb{A}^1 \subset \mathbb{A}^3$$

in affine space, the simplest (Calabi–Yau) threefold of all. Here is a summary of the contents of this work.

The first two chapters contain the dictionary and the main tools and theorems we will be using throughout, but no original results. After introducing DT and PT invariants, we define the ring of motivic weights and the central notion of *virtual motive* of a scheme; we compute virtual motives for the three-loop quiver as an example to illustrate the technique used later on.

The third chapter is a joint work with Martin G. Gulbrandsen in which we compute the Euler characteristic of the generalized Kummer scheme of an abelian threefold. The formula was conjectured by Gulbrandsen in a previous paper, and allows one to compute Gulbrandsen's version of the degree zero DT invariants of an abelian threefold, which unlike the classical ones are nonzero.

The fourth chapter is the content of an independent paper, in which we compute the virtual Euler characteristic of the "Quot scheme of n points" of the ideal sheaf of a curve in a threefold. For a *rigid* smooth curve in a Calabi–Yau threefold, this calculation is equivalent to a "local DT/PT correspondence" at C. We conjecture the correspondence to hold for all smooth curves and we prove this is indeed the case in Chapter 5.

The fifth chapter contains the proof of the DT/PT correspondence for arbitrary smooth curves in Calabi–Yau threefolds. We exploit results from the previous chapter, along with a local study of the Hilbert–Chow morphism.

The sixth chapter proves that the Quot scheme of n points of the ideal of a line in $\mathbb{A}^3$, is a global critical locus, just like the Hilbert scheme of points of $\mathbb{A}^3$. This gives a canonical virtual motive for this Quot scheme.

The seventh chapter applies two different strategies to compute the motivic partition function of the Quot scheme of a line in three-space. The result is not entirely explicit, but we conjecture an explicit formula in Chapter 8. We can, however, *define* a virtual motive for the Quot scheme of an arbitrary smooth curve embedded in a smooth quasi-projective threefold. Given the calculations of Chapters 4 and 5, this provides many examples of motivic DT invariants in the projective case.

The eighth chapter contains a conjectural explicit formula for the virtual motive of the Quot scheme constructed in Chapter 6. We verify the formula by hand up to 4 points.

Part I

PRELIMINARIES

1 | GEOMETRIC TOOLS

1.1 Moduli spaces

Let Y be a nonsingular, complex projective threefold. Fix an integer m and a homology class $\beta \in H_2(Y,\mathbb{Z})$. The main character of Donaldson–Thomas (DT for short) theory is the moduli space of *ideal sheaves*

$$I_m(Y,\beta) = \left\{ \mathscr{I}_Z \subset \mathscr{O}_Y \,\middle|\, \chi(\mathscr{O}_Z) = m,\, [Z] = \beta \right\},$$

which is canonically isomorphic to the Hilbert scheme of subschemes $Z \subset Y$ of codimension at least 2 [62, Section 2]. The main character of Pandharipande–Thomas (PT for short) theory is the moduli space of *stable pairs*,

$$P_m(Y,\beta) = \left\{ \mathscr{O}_Y \xrightarrow{s} F \,\middle|\, \begin{array}{c} F \text{ is pure, } \dim F = 1, \dim(\operatorname{coker} s) = 0, \\ \chi(F) = m,\, [\operatorname{Supp} F] = \beta \end{array} \right\}.$$

The Hilbert scheme and the moduli space of stable pairs are isomorphic along the open subscheme parametrizing Cohen–Macaulay curves. Curves with isolated points are routine in DT theory, but strictly forbidden in PT theory (the cokernel of the section $s : \mathscr{O}_Y \to F$ is supported on the Cohen–Macaulay curve $\operatorname{Supp} F \subset Y$), which might explain why the PT moduli space is usually easier to handle than the Hilbert scheme. The DT and PT moduli spaces carry a perfect obstruction theory of virtual dimension

$$d_\beta = \int_\beta c_1(Y).$$

See [8, 9] for foundations on perfect obstruction theories and virtual fundamental classes. The virtual dimension vanishes in the Calabi–Yau case, when $c_1(Y) = 0$. Each perfect obstruction theory gives canonically a virtual fundamental class living in the Chow group $A_{d_\beta} \to H_{2d_\beta}$ of the moduli space. When $d_\beta > 0$ insertions are required in order to extract enumerative invariants. These will always be integers. When $d_\beta = 0$, the (DT, PT) invariants of Y are defined as the degree of the associated 0-cycles classes,

$$\mathsf{DT}^Y_{m,\beta} = \int_{[I_m(Y,\beta)]^{\mathrm{vir}}} 1, \qquad \mathsf{PT}^Y_{m,\beta} = \int_{[P_m(Y,\beta)]^{\mathrm{vir}}} 1.$$

1.1.1 The Behrend function

We now briefly recall why DT and PT invariants, unlike the (rational) GW invariants, can be computed "motivically". Let $\mathcal{C}_X$ be the group of constructible functions on a complex scheme X. The local Euler obstruction is a well-studied group isomorphism

$$\mathrm{Eu}: Z_*X \xrightarrow{\sim} \mathcal{C}_X.$$

Behrend [5] defined the *distinguished cycle* $\mathfrak{c}_X$ of X by considering the signed support of the intrinsic normal cone of X. We recall a couple of definitions from [5].

Definition 1.1.1. Let X be a complex scheme. The *Behrend function* of X is

$$\nu_X = \mathrm{Eu}(\mathfrak{c}_X) \in \mathcal{C}_X. \qquad \diamondsuit$$

Definition 1.1.2. The *virtual* (or *weighted*) *Euler characteristic* of a complex scheme X is the integer

$$\chi_{\mathrm{vir}}(X) = \int_X \nu_X \,\mathrm{d}\chi = \sum_{n\in\mathbb{Z}} n\chi(\nu_X^{-1}(n)). \qquad \diamondsuit$$

THEOREM 1.1.3 ([5, Thm. 4.18]). *Let X be a proper scheme equipped with a symmetric perfect obstruction theory. Then*

$$\int_{[X]^{\mathrm{vir}}} 1 = \chi_{\mathrm{vir}}(X).$$

In particular, the "virtual count" of a proper scheme X does not depend on the chosen symmetric perfect obstruction theory. The theorem implies that DT and PT invariants of a Calabi–Yau threefold Y can be computed via cut-and-paste techniques as the virtual Euler characteristic of the moduli space. Sometimes we will write $\tilde{\chi}$ instead of χ_{vir}. We will see the most important properties of the Behrend function in action in Section 4.4.

Remark 1.1.4. Gromov–Witten theory is not motivic: the obstruction theory on the moduli space of stable maps $\overline{M}_g(Y,\beta)$ is only symmetric over the open (possibly empty) locus of maps which are immersions of a smooth curve. $\diamondsuit$

1.2 Critical loci

We sketch the well-known fact that a critical locus has a natural perfect symmetric obstruction theory. There is a natural motivic incarnation of the induced virtual fundamental class, which will be recalled in Section 2.1.3.

Definition 1.2.1. A critical locus is a complex scheme Z of the form $Z(\mathrm{d}f)$, where $f: V \to \mathbb{A}^1$ is a regular function on a smooth scheme V. $\diamondsuit$

For moduli spaces of interest in sheaf counting, to be a global critical locus is quite a restrictive condition. However, besides the obvious example of smooth schemes, there are the following examples, all coming more or less directly from moduli of quiver representations:

- the Hilbert scheme of points $\mathrm{Hilb}^n(\mathbb{A}^3)$, cf. Example 2.3.5;

- The moduli space of stable pairs $P_m(X,\ell[\mathbb{P}^1])$ on the *resolved conifold* X, namely the total space of the rank two bundle $\mathscr{O}_{\mathbb{P}^1}(-1,-1)$ over $\mathbb{P}^1$. For a proof see for instance [76, Thm. 3.2]. This critical locus is obtained by considering moduli of framed representations of the conifold quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "y_1"] \arrow[r, bend left=15, "x_1"] & \bullet \arrow[l, bend left=15, "x_2"] \arrow[l, bend left=50, "y_2"] \end{tikzcd}$$

  with respect to the potential $W = x_1(x_2 y_1 y_2 - y_2 y_1 x_2)$;

- The Hilbert scheme $I_m(X,\ell[\mathbb{P}^1])$, where X is again the resolved conifold. This can be inferred from the work of Nagao–Nakajima [56, Sections 2, 3].[1]

Let (V,f,Z) be a critical locus as in Definition 1.2.1, and let $d = \dim V$. If $\mathscr{I} \subset \mathscr{O}_V$ is the ideal of $Z \subset V$ and we shorten $s = \mathrm{d}f$, the cosection $s^\vee : T_V \to \mathscr{O}_V$ necessarily factors through $\mathscr{I}$, hence we can restrict it to Z to get a surjection

$$s^\vee\big|_Z : T_V\big|_Z \twoheadrightarrow \mathscr{I}/\mathscr{I}^2.$$

Composing the natural inclusion $C_{Z/V} \subset N_{Z/V}$ with the closed immersion

$$\operatorname{Spec}\operatorname{Sym} s^\vee\big|_Z : N_{Z/V} \to \Omega_V\big|_Z,$$

we embed the normal cone $C_{Z/V}$ as a d-dimensional subvariety of the rank d bundle $\Omega_V|_Z$. This embedding can be seen as a symmetric perfect obstruction theory on Z in the sense of [9]. The associated virtual fundamental class is the zero-dimensional cycle class

$$[Z]^{\mathrm{vir}} = 0^*[C_{Z/V}] \in A_0(Z), \tag{1.2.1}$$

where $0^* : A_d(\Omega_V|_Z) \xrightarrow{\sim} A_0(Z)$ is the inverse of the flat pullback isomorphism.

1.2.1 Vanishing cycles

Let (V,f,Z) be as in Definition 1.2.1. Notice that $Z = V(\mathrm{d}f)$ is the singular locus of the central fibre $V_0 = f^{-1}(0)$. For every point $x \in Z$ one can find a small enough $0 < \epsilon < 1$ such that the restriction $V^\times_{\epsilon,\eta} = B_\epsilon(x) \cap f^{-1}(\Delta^\times_\eta) \to \Delta^\times_\eta$ is a topological fibration for $0 < \eta \ll \epsilon < 1$. This is called the *Milnor fibration*, and its fibre $F_{f,x}$ is called the *Milnor fibre* of f at x. All this can be summarized in the classical picture

$$\begin{tikzcd} Z \arrow[r, hook] & V_0 \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "f"] & V^\times_{\epsilon,\eta} \arrow[l, hook', "j"'] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[l, "p"'] \arrow[d] \\ & \{0\} \arrow[r, hook] & \mathbb{A}^1 & \Delta^\times_\eta \arrow[l, hook'] & \widetilde{\Delta^\times_\eta} \arrow[l] \end{tikzcd}$$

1 We thank Balázs Szendrői for helping us identifying the right reference.

and from here the *nearby cycle* functor $\psi_f : D_c^b(V) \to D_c^b(V_0)$ is defined as

$$\psi_f \mathscr{F}^\cdot = i^{-1} R(j \circ p)_*(j \circ p)^* \mathscr{F}^\cdot.$$

Here D_c^b denotes the bounded derived category of sheaves with constructible cohomology. The functor $\phi_f : D_c^b(V) \to D_c^b(V_0)$ of *vanishing cycles* is defined as follows: the complex $\phi_f \mathscr{F}^\cdot$ is the cone of the adjunction map $i^{-1}\mathscr{F}^\cdot \to \psi_f \mathscr{F}^\cdot$. The nearby cycle complex and the vanishing cycle complex of f are defined as

$$\Psi_f = \psi_f \underline{\mathbb{C}}_V, \qquad \Phi_f = \phi_f \underline{\mathbb{C}}_V.$$

The critical locus Z supports the vanishing cycles (the Milnor fibre at a smooth point $x \in V_0 \setminus Z$ is contractible), and Φ_f computes the reduced cohomology of the Milnor fibre, in the sense that

$$\mathcal{H}^i(\Phi_f)_x \cong \widetilde{H}^i(F_{f,x}, \mathbb{C}).$$

Let ν_Z be the Behrend function of $Z = Z(\mathrm{d} f)$. The value $\nu_Z(x)$ is the "contribution" of $x \in Z$ to the virtual Euler characteristic $\chi_{\mathrm{vir}}(Z)$. It is a deep result [65, Cor. 2.4 (iii)] that ν_Z equals the *Milnor function* of f, the function $\mu_f : Z \to \mathbb{Z}$ counting the "number of vanishing cycles". The latter is defined by

$$\mu_f(x) = (-1)^d(1 - \chi(F_{f,x})), \tag{1.2.2}$$

where as before $d = \dim V$. The value $\mu_f(x)$ is sometimes called the *Hodge spectrum* of f at x. For instance, when $f = 0$, we have $Z = V$ and $\nu_Z \equiv (-1)^d$. Granting the identity $\nu_Z = \mu_f$, one can write

$$\begin{aligned}
\nu_Z(x) &= (-1)^{d-1}(\chi(F_{f,x}) - 1) \\
&= (-1)^{d-1} \sum (-1)^i \dim \widetilde{H}^i(F_{f,x}, \mathbb{C}) \\
&= (-1)^{d-1} \sum (-1)^i \dim \mathcal{H}^i(\Phi_f)_x \\
&= (-1)^{d-1} \chi\left(\Phi_f\big|_x\right).
\end{aligned}$$

This can be compactly rewritten as

$$\nu_Z = (-1)^{d-1}\chi(\Phi_f) = \chi(\Phi_f[d-1]). \tag{1.2.3}$$

Aside 1.2.1. Formula (1.2.3) is the moral responsible for Donaldson–Thomas invariants to "look like" Euler characteristics. In fact, a moduli space M of simple coherent sheaves (or complexes) on a Calabi–Yau threefold is, locally around every closed point $p \in M$, isomorphic to a critical locus. This is a hard result [13, 10]. It can be proven [12, 42] that the sheaves of vanishing cycles Φ_f on the critical charts glue to a global perverse sheaf Φ on M, whose Euler characteristic computes the DT invariant,

$$\int_M \nu_M \, \mathrm{d}\chi = \sum_{i \geq 0} (-1)^i h^i(M, \Phi).$$

We refer to [77, Section 4] for a thorough definition of the *cohomological DT invariant* $H^*(M, \Phi)$.

2 | MOTIVIC TOOLS

2.1 Grothendieck rings of varieties

All schemes are defined over $\mathbb{C}$. Most of the material covered in this section can be generalized to arbitrary fields, see [25] and [49] for nice surveys on the subject. The conventions we will adopt later for our motivic computations are those of [7]. We recall them here for completeness.

Definition 2.1.1. Let S be a variety over $\mathbb{C}$.

(i) The *Grothendieck group of S-varieties* is the free abelian group $K_0(\mathsf{Var}_S)$ generated by isomorphism classes $[X]$ of S-varieties $X \to S$, modulo the scissor relations, namely the identities $[Y] = [X] + [Y \setminus X]$ whenever X is a closed S-subvariety of Y. The group $K_0(\mathsf{Var}_S)$ is a ring via $[Y] \cdot [Z] = [Y \times_S Z]$.

(ii) We denote by $\mathbb{L} = [\mathbb{A}^1_S] \in K_0(\mathsf{Var}_S)$ the *Lefschetz motive*, the class of the affine line over S. ♢

The class $[X] \in K_0(\mathsf{Var}_{\mathbb{C}})$ of a variety X is called its *motive*, or universal Euler characteristic. We write $[X]_S$ when we wish to emphasize the base scheme. Given a morphism $f : S \to T$ of complex varieties, we have an induced *pull-back* map

$$f^* : K_0(\mathsf{Var}_T) \to K_0(\mathsf{Var}_S)$$

which is a ring homomorphism given by $f^*[X] = [X \times_T S]$ on generators. In particular, $K_0(\mathsf{Var}_S)$ is a $K_0(\mathsf{Var}_{\mathbb{C}})$-module. Composition with f also gives a *direct image* homomorphism $f_! : K_0(\mathsf{Var}_S) \to K_0(\mathsf{Var}_T)$, which is $K_0(\mathsf{Var}_T)$-linear. The ring

$$\mathcal{M}_S = K_0(\mathsf{Var}_S)\big[\mathbb{L}^{-1/2}\big]$$

is called the *ring of motivic weights*. The above maps extend to a ring homomorphism $f^* : \mathcal{M}_T \to \mathcal{M}_S$ and an $\mathcal{M}_T$-linear map $f_! : \mathcal{M}_S \to \mathcal{M}_T$.

Definition 2.1.2. A morphism of schemes $f : Y \to X$ is a *Zariski fibration* if there is a Zariski open cover $X = \bigcup_i X_i$ and isomorphisms $f^{-1}(X_i) \xrightarrow{\sim} X_i \times F_i$ over X_i. ♢

When f is a Zariski fibration with fibres all isomorphic to a typical fibre F, we will simply say f has fibre F. The most important tools for computations in the Grothendieck ring, which we will use extensively, are the following:

- if $Y \to X$ is a bijective morphism of varieties, then $[X] = [Y]$ in $K_0(\mathsf{Var}_{\mathbb{C}})$;

- if $Y \to X$ is a Zariski fibration with fibre F, then $[Y] = [X] \cdot [F]$.

One can also define Grothendieck rings of schemes and algebraic spaces. These are both isomorphic to $K_0(\mathsf{Var}_{\mathbb{C}})$ by [16, Lemma 2.12]. The situation is different with stacks. There is a Grothendieck ring of stacks

$$K_0(\mathsf{St}_{\mathbb{C}}),$$

generated by isomorphism classes of stacks of finite type over $\mathbb{C}$, having affine geometric stabilizers. We refer the reader to [26] or to [16, Definition 3.6] for the precise definition. Here we simply recall that $K_0(\mathsf{St}_{\mathbb{C}})$ can be obtained from $K_0(\mathsf{Var}_{\mathbb{C}})$ in the following equivalent ways:

- by localizing at the classes of special algebraic groups,
- by localizing at $\mathbb{L}$ and $\mathbb{L}^i - 1$ for $i \geq 1$,
- by localizing at the classes $[\mathrm{GL}_d]$ for $d \geq 1$.

The motivic class of a quotient stack U/G is the quotient $[U]/[G]$ when G is *special*, but not in general. See [26] or [16, Lemmas 3.8 and 3.9] for a proof of this fact.

Example 2.1.3. We will let GL_d denote the class of GL_d in $K_0(\mathsf{Var}_{\mathbb{C}})$ throughout. As proved in [16, Lemma 2.6], one has

$$\mathsf{GL}_d = \prod_{i=0}^{d-1} (\mathbb{L}^d - \mathbb{L}^i) = \mathbb{L}^{\binom{d}{2}} \cdot \prod_{k=1}^{d} (\mathbb{L}^k - 1).$$

Sometimes, one uses the shorthand $[d]_{\mathbb{L}}! = \prod_{k=1}^{d} (\mathbb{L}^k - 1)$. Then, the motive of the Grassmannian can be computed as

$$\big[\mathrm{Gr}(k,n)\big] = \frac{[n]_{\mathbb{L}}!}{[k]_{\mathbb{L}}![n-k]_{\mathbb{L}}!} \in K_0(\mathsf{Var}_{\mathbb{C}}). \tag{2.1.1}$$

$\diamondsuit$

The commuting variety and the Feit–Fine formula

We give an example of motivic classes in the Grothendieck ring of stacks that will be important later on. Let V be an n-dimensional complex vector space and let

$$C_n = \big\{ (A,B) \in \mathrm{End}(V)^2 \,\big|\, [A,B] = 0 \big\} \subset \mathrm{End}(V)^2 \tag{2.1.2}$$

be the commuting variety. Letting GL_n act on C_n by simultaneous conjugation, one can form the quotient stack

$$\mathcal{C}(n) = C_n / \mathrm{GL}_n,$$

which is equivalent to the stack $\mathsf{Coh}_n(\mathbb{A}^2)$ of finite coherent sheaves of length n on the affine plane. Letting

$$\widetilde{c}_n = \big[\mathcal{C}(n)\big] = \frac{[C_n]}{\mathsf{GL}_n} \in K_0(\mathsf{St}_{\mathbb{C}}) \tag{2.1.3}$$

be the motivic class of the stack $\mathcal{C}(n)$,[1] let us form the generating series

$$\mathsf{C}(t)=\sum_{n\geq 0}\widetilde{c}_n t^n \in K_0(\mathsf{St}_{\mathbb{C}})[\![t]\!].$$

The next result is a formula essentially due to Feit and Fine, but also proven recently by Behrend–Bryan–Szendrői and Bryan–Morrison.

THEOREM 2.1.4 ([28, 7, 19]). *One has the formula*

$$\mathsf{C}(t)=\prod_{k\geq 1}\prod_{m\geq 1}(1-\mathbb{L}^{2-k}t^m)^{-1}.$$

Aside 2.1.1. It has been known since a long time that the variety of pairs of commuting matrices C_n is irreducible [54, 67]. The same is true for the space $N_n \subset C_n$ of *nilpotent* commuting linear operators, see [2] for a proof in characteristic zero and [3] for an extension to fields of characteristic bigger than $n/2$. Premet even showed irreducibility of N_n over *any* field [66].

2.1.1 Equivariant Grothendieck rings

Let G be a finite group.

Definition 2.1.5. A G-action on a variety X is said to be *good* if every point of X has a G-invariant affine open neighborhood. ♢

Actions on quasi-projective varieties are good. Moreover, for a good G-action, an orbit space X/G exists at least as an algebraic space.

Definition 2.1.6. Let S be a variety with good G-action. We let $\widetilde{K}_0^G(\mathsf{Var}_S)$ be the abelian group generated by isomorphism classes $[X,G]$ of S-varieties with good action, modulo the G-scissor relation (over S). We define the *equivariant Grothendieck group* $K_0^G(\mathsf{Var}_S)$ by further quotienting out the relations

$$[V,G]=[\mathbb{A}^r_S],$$

whenever $V\to S$ is a G-equivariant vector bundle of rank r. The right hand side is taken with the trivial G-action. ♢

There is a natural ring structure on $K_0^G(\mathsf{Var}_S)$ given by fibre product. If the G-action on S is trivial, $\widetilde{K}_0^G(\mathsf{Var}_S)$ becomes a $K_0(\mathsf{Var}_S)$-algebra and there exists a natural $K_0(\mathsf{Var}_S)$-linear "quotient map"

$$\pi_G:\widetilde{K}_0^G(\mathsf{Var}_S)\to K_0(\mathsf{Var}_S) \tag{2.1.4}$$

given on generators by taking the orbit space. A similar story is true for the rings

$$\widetilde{\mathcal{M}}_S^G=\widetilde{K}_0^G(\mathsf{Var}_S)[\mathbb{L}^{-1/2}]$$
$$\mathcal{M}_S^G=K_0^G(\mathsf{Var}_S)[\mathbb{L}^{-1/2}]$$

1 The notation $\widetilde{c}_n$ is as in [7, Section 2].

which we refer to as rings of *equivariant motivic weights*. When the G-action on S is trivial these rings become $\mathcal{M}_S$-algebras

$$\mathcal{M}_S \to \widetilde{\mathcal{M}}_S^G \twoheadrightarrow \mathcal{M}_S^G,$$

and (2.1.4) extends to an $\mathcal{M}_S$-linear quotient map

$$\pi_G : \widetilde{\mathcal{M}}_S^G \to \mathcal{M}_S. \tag{2.1.5}$$

The map π_G extends to a *ring* homomorphism $\mathcal{M}_S^G \to \mathcal{M}_S$ when G is finite abelian, but not in general. The following result will be used in Section 7.3.3.

LEMMA 2.1.7 ([7, Lemma 2.4]). *For any $n > 0$ there exists a n-th power map*

$$(-)^n : \mathcal{M}_{\mathbb{C}} \to \widetilde{\mathcal{M}}_{\mathbb{C}}^{S_n}$$

defined by the property that for $A \in \mathcal{M}_{\mathbb{C}}$ representing a quasi-projective variety, A^n is the class of the n-th power of that variety, carrying the standard S_n-action.

The monodromic motivic ring

Let $\mu_n = \operatorname{Spec} \mathbb{C}[x]/(x^n - 1)$ be the group of n-th roots of unity. One can define good actions of the procyclic group

$$\hat{\mu} = \varprojlim \mu_n$$

as actions that factor through a good μ_n-action for some n. The additive group $\mathcal{M}_S^{\hat{\mu}}$ also carries a commutative bilinear associative product $\star$ called the *convolution product*. See [25, Section 5] or [49, Section 7] for its definition. The product $\star$ provides an alternative ring structure on the group of $\hat{\mu}$-equivariant motivic weights, and restricts to the usual product on the subring

$$\mathcal{M}_S \subset \mathcal{M}_S^{\hat{\mu}}$$

of classes with trivial $\hat{\mu}$-action. The main role of $\star$ will be played through the motivic Thom–Sebastiani theorem, cf. Theorem 2.1.17.

2.1.2 Motivic measures

Quoting Looijenga [49],

> "The ring $\mathcal{M}_k$ is interesting, big, and hard to grasp. Fortunately, there are several characteristics of $\mathcal{M}_k$ (i. e. ring homomorphisms from $\mathcal{M}_k$ to a ring) that are well understood."

Ring homomorphisms with source $K_0(\mathsf{Var}_{\mathbb{C}})$ or $\mathcal{M}_{\mathbb{C}}$ are frequently called *motivic measures*, realizations, or generalized Euler characteristics. We recall some of them here. Fix $S = \operatorname{Spec} \mathbb{C}$.

Let $K_0(\mathsf{HS})$ be the Grothendieck ring of the abelian category HS of Hodge structures. The *Hodge characteristic* of a complex variety X, defined as

$$\chi_h(X) = \sum_{i \geq 0} (-1)^i \left[H_c^i(X, \mathbb{Q})\right] \in K_0(\mathsf{HS}),$$

is a motivic measure. The E*-polynomial* is the specialization

$$E(X) = \sum_{p,q\geq 0} (-1)^{p+q} h^{p,q}\big(H_c^{p+q}(X,\mathbb{Q})\big) u^p v^q \in \mathbb{Z}[u,v].$$

As $E(\mathbb{A}^1_{\mathbb{C}}) = uv$, the E-polynomial can be extended to a motivic measure

$$E : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}\big[u,v,(uv)^{-1/2}\big]$$

satisfying $E(\mathbb{L}^{1/2}) = (uv)^{1/2}$. Following the conventions in [7], the further specialization

$$u = v = -q^{1/2}, \qquad (uv)^{1/2} = q^{1/2}$$

defines the *weight polynomial* $W : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[q^{\pm 1/2}]$ and one has $W(\mathbb{L}) = q$. The further specialization $q^{1/2} = -1$ recovers the Euler characteristic

$$\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z},$$

extending $\chi : K_0(\mathsf{Var}_{\mathbb{C}}) \to \mathbb{Z}$. There is a natural extension [25, Section 2] to a ring homomorphism

$$\chi : \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \to \mathbb{Z}.$$

The following definition will be central for us.

Definition 2.1.8. A *virtual motive* of a complex scheme X is a class $\zeta \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ such that $\chi(\zeta) = \chi_{\mathrm{vir}}(X)$. When X is a moduli space of sheaves on a Calabi–Yau threefold, a virtual motive for X will be called a *motivic Donaldson–Thomas invariant*. ♢

Remark 2.1.9. Motivic DT invariants can be nonzero when the numerical DT invariants vanish. An example is the 0-dimensional DT theory of an abelian threefold Y, which is trivial since

$$\chi_{\mathrm{vir}}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y) = 0 \ \text{ for } n > 0,$$

but the refinement $\zeta = [\mathrm{Hilb}^n Y]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ defined in [7] is nontrivial. ♢

2.1.3 The virtual motive of a critical locus

Let V be a smooth scheme of dimension d, and let $f : V \to \mathbb{A}^1$ be a regular function with zero scheme V_0 and critical locus $Z \subset V_0$. We next recall the definition of the canonical virtual motive $[Z]_{\mathrm{vir}}$ attached to the pair (V,f). Roughly speaking, to refine the numerical identity

$$\nu_Z = -(-1)^d \chi(\Phi_f)$$

obtained in (1.2.3) to a motivic setting, we are going to replace "-1" with $\mathbb{L}^{-1/2}$ and $\Phi_f \in D_c^b(Z)$ with $[\phi_f]_Z \in \mathcal{M}_Z^{\hat{\mu}}$, the relative class of *motivic vanishing cycles*. For completeness, we wish to recall the definition of this class, due to Denef and Loeser.

Let $n \geq 1$ be an integer, and let $J_n V$ be the space of n-arcs (also known as n-jet scheme) on the smooth variety V. Its complex points are

$$J_n V = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}\, \mathbb{C}[t]/t^{n+1}, V).$$

We are interested in smaller arc spaces, namely

$$\mathfrak{X}_n = \left\{ \gamma \in J_n V \,\middle|\, \mathrm{ord}_t\, f \circ \gamma(t) = n \right\} \subset J_n V,$$

and the even smaller space

$$\mathfrak{X}_{n,1} = \left\{ \gamma \in J_n V \,\middle|\, f \circ \gamma(t) \equiv t^n \,(\mathrm{mod}\, t^{n+1}) \right\} \subset \mathfrak{X}_n.$$

Under the truncation map $J_n V \to V$, the space $\mathfrak{X}_n$ is mapped inside V_0, and this in particular makes $\mathfrak{X}_{n,1}$ into a V_0-variety. Moreover, the natural $\mathbb{G}_m$-action on $\mathfrak{X}_n$ restricts to a good μ_n-action on $\mathfrak{X}_{n,1}$, so we may consider the relative equivariant classes

$$\left[\mathfrak{X}_{n,1}, \hat{\mu}\right]_{V_0} \in \mathcal{M}_{V_0}^{\hat{\mu}}.$$

Definition 2.1.10. The power series

$$\mathsf{Z}_f(T) = \sum_{n \geq 1} \left[\mathfrak{X}_{n,1}, \hat{\mu}\right]_{V_0} \mathbb{L}^{-dn} T^n \in \mathcal{M}_{V_0}^{\hat{\mu}}[\![T]\!]$$

is called the *motivic zeta function* of f. ♢

The motivic zeta function is an intrinsic invariant of a regular function. Denef and Loeser proved its rationality over any field of characteristic zero, by means of an explicit formula in terms of an embedded resolution [25, Thm. 3.3.1].[2] For any point $x \in V_0$, there is a "fibre map"

$$\mathrm{Fib}_x : \mathcal{M}_{V_0}^{\hat{\mu}} \to \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

defined on generators by $[Y, \hat{\mu}] \mapsto [Y \times_{V_0} k(x), \hat{\mu}]$.

Definition 2.1.11 ([25, Section 3])**.** Given $f : V \to \mathbb{A}^1$ as above,

(i) $\mathcal{S}_f = [\psi_f]_{V_0} = -\lim_{T\to\infty} \mathsf{Z}_f(T) \in \mathcal{M}_{V_0}^{\hat{\mu}}$ is called the *relative motivic nearby fibre*;

(ii) $[\phi_f]_{V_0} = [\psi_f]_{V_0} - 1 \in \mathcal{M}_{V_0}^{\hat{\mu}}$ is called the *relative motivic vanishing cycle* (here $1 = [V_0]_{V_0}$ is the ring identity);

(iii) $\mathcal{S}_{f,x} = \mathrm{Fib}_x(\mathcal{S}_f)$ is called the *motivic Milnor fibre* of f at x. ♢

As $[\phi_f]_{V_0}$ vanishes over the smooth locus of V_0, the relative motivic vanishing cycle is a relative class

$$[\phi_f]_Z \in \mathcal{M}_Z^{\hat{\mu}}$$

living on the singular locus $Z \subset V_0$ (analogously to the complex $\Phi_f \in D_c^b(V_0)$, which is supported on Z). We will let

$$[\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

denote its pushforward under the structure morphism $Z \to \mathrm{Spec}\, \mathbb{C}$.

2 Denef and Loeser actually work in the ring $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$, which we could have done, too. What is crucial to obtain rationality of $\mathsf{Z}_f(T)$ is having $\mathbb{L}$ inverted, not a square root of it.

Remark 2.1.12. If we let $\mathfrak{X}_{n,1}(x) \subset \mathfrak{X}_{n,1}$ be the space of arcs based at x, and we form the generating series $\mathsf{Z}_{f,x}(T) = \sum_{n\geq 1}[\mathfrak{X}_{n,1}(x)]T^n$, one can compute the Euler characteristic of the Milnor fibre as

$$\chi(F_{f,x}) = -\lim_{T\to\infty}\chi(\mathsf{Z}_{f,x}(T)). \qquad \diamondsuit$$

Definition 2.1.13 ([7, Def. 2.14])**.** The *relative virtual motive* of $Z = Z(\mathrm{d}f)$ attached to $f: V \to \mathbb{A}^1$ is the class

$$[Z]_{\mathrm{relvir}} = -\mathbb{L}^{-d/2}[\phi_f]_Z \in \mathcal{M}_Z^{\hat{\mu}},$$

where $d = \dim V$. The *absolute virtual motive* of Z is the pushforward of this class to a point, namely

$$[Z]_{\mathrm{vir}} = -\mathbb{L}^{-d/2}[\phi_f] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \qquad \diamondsuit$$

Example 2.1.14. When $f = 0$, the smooth scheme $Z = V$ has virtual motive

$$[V]_{\mathrm{vir}} = \mathbb{L}^{-d/2}[V] \in \mathcal{M}_{\mathbb{C}},$$

as $[\psi_f]_V = 0$ in this case. $\diamondsuit$

The class $[Z]_{\mathrm{vir}}$ just defined is a virtual motive in the sense of Definition 2.1.8. Indeed, the fibrewise Euler characteristic of $[Z]_{\mathrm{relvir}}$ at $x \in Z$ is precisely

$$-(-1)^d\chi\left(\Phi_f\big|_x\right) = \nu_Z(x).$$

When $Z = Z(\mathrm{d}f)$ is proper, the virtual motive $[Z]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ relates to the virtual class $[Z]^{\mathrm{vir}} \in A_0(Z)$ defined in (1.2.1) through Theorem 1.1.3,

$$\int_{[Z]^{\mathrm{vir}}} 1 = \chi_{\mathrm{vir}}(Z) = \chi([Z]_{\mathrm{vir}}).$$

For future use in Sections 6.2.2 and 7.3.1, we reproduce here from [7, Theorem B.1] a statement determining the virtual motive of a critical locus attached to a family with "nice" equivariance properties. We need a definition.

Definition 2.1.15. Let X be a variety, $f: X \to \mathbb{A}^1$ a regular function, $\mathbf{T}$ a connected complex torus acting on X. We say that f is $\mathbf{T}$-equivariant with respect to a character $\chi: \mathbf{T} \to \mathbb{G}_m$ if $f(t\cdot x) = \chi(t)\cdot f(x)$ for all $t \in \mathbf{T}$ and $x \in X$. An action of $\mathbb{G}_m$ on X is said to be *circle compact* if it has compact fixed locus, and if limits $\lim_{t\to 0} t\cdot x$ exist for all $x \in X$. $\diamondsuit$

THEOREM 2.1.16. *Let $f: X \to \mathbb{A}^1$ be a regular function on a smooth complex quasi-projective variety, with critical locus Z. Assume X is acted on by a connected complex torus $\mathbf{T}$ in such a way that f is $\mathbf{T}$-equivariant with respect to a primitive character $\chi: \mathbf{T} \to \mathbb{G}_m$.*

(i) *If there is a one parameter subgroup $\mathbb{G}_m \subset \mathbf{T}$ such that the induced action is circle compact, then*

$$[\phi_f] = [f^{-1}(1)] - [f^{-1}(0)] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

(ii) *Let* $\mathsf{a} : Z \to Z_{\mathrm{aff}}$ *be the affinization of* Z*. If, in addition to the assumption in (i), the hypersurface* $X_0 \subset X$ *is reduced, then the relative class* $[\phi_f]_{Z_{\mathrm{aff}}} = \mathsf{a}_![\phi_f]_Z$ *lies in the subring* $\mathcal{M}_{Z_{\mathrm{aff}}}$ *of classes with trivial monodromy.*

As explained in [7, Section 2.7], families $f : X \to \mathbb{A}^1$ that are $\mathbf{T}$-equivariant with respect to a primitive character $\chi : \mathbf{T} \to \mathbb{G}_m$ are trivial away from the central fibre. Because χ is primitive, one can find a 1-parameter subgroup $j : \mathbb{G}_m \hookrightarrow \mathbf{T}$ such that $\chi \circ j$ is an isomorphism. This implies that the action $(\lambda, x) \mapsto \lambda \cdot x$ by the $\mathbb{G}_m$ subgroup induces an isomorphism

$$X_1 \times \mathbb{G}_m \xrightarrow{\sim} X \setminus X_0, \tag{2.1.6}$$

whose inverse is given by $x \mapsto (f(x)^{-1} \cdot x, f(x))$. Here X_1 denotes the "generic fibre" $f^{-1}(1)$.

We end this section by recalling the *motivic Thom–Sebastiani theorem.*

THEOREM 2.1.17 (Motivic Thom–Sebastiani [24, 49]). *Let* $f : X \to \mathbb{A}^1$ *and* $g : Y \to \mathbb{A}^1$ *be regular functions on smooth varieties* X *and* Y*. Consider the function* $f \oplus g : X \times Y \to \mathbb{A}^1$ *given by* $(x, y) \mapsto f(x) + g(y)$*. Let* $i : X_0 \times Y_0 \to (X \times Y)_0$ *be the inclusion, and let* p_X *and* p_Y *be the projections from* $X_0 \times Y_0$*. Then one has*

$$i^*[\phi_{f\oplus g}]_{(X\times Y)_0} = p_X^*[\phi_f]_{X_0} \star p_Y^*[\phi_g]_{Y_0} \in \mathcal{M}^{\hat{\mu}}_{X_0\times Y_0}.$$

2.2 Power structures

Let R be a commutative unitary ring. We recall the notion of a *power structure* on R, mainly following [36, 37].

Definition 2.2.1. A *power structure* on R is a map

$$\begin{gathered}(1 + tR[\![t]\!]) \times R \to 1 + tR[\![t]\!] \\ (A(t), X) \mapsto A(t)^X\end{gathered}$$

satisfying the following conditions:

- $A(t)^0 = 1$
- $A(t)^1 = A(t)$
- $(A(t) \cdot B(t))^X = A(t)^X \cdot B(t)^X$
- $A(t)^{X+Y} = A(t)^X \cdot A(t)^Y$
- $A(t)^{XY} = (A(t)^X)^Y$
- $(1 + t)^X = 1 + Xt + O(t^2)$
- $A(t)^X\big|_{t\to t^k} = A(t^k)^X$. ◊

Before introducing the power structure on the Grothendieck ring of varieties, let us revisit the combinatorial formula expressing the m-th power (m being a natural number) of a power series with coefficients A_n in a $\mathbb{Q}$-algebra, namely

$$\left(1+\sum_{n>0} A_n t^n\right)^m = 1+\sum_{\alpha}\left(\prod_{i=0}^{\|\alpha\|-1}(m-i)\cdot\frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!}\right)t^{|\alpha|}. \tag{2.2.1}$$

The sum on the right is indexed by partitions $\alpha=(1^{\alpha_1}\cdots i^{\alpha_i}\cdots \ell^{\alpha_\ell})$, and we have set

$$\|\alpha\| = \sum_i \alpha_i, \qquad |\alpha| = \sum_i i\alpha_i.$$

The latter is the *size* of α. Let us now focus on $K_0(\mathrm{Var}_{\mathbb{C}})$. If X is a variety and $A(t)=1+\sum_{n>0} A_n t^n$ is a power series in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$, setting

$$A(t)^{[X]} = 1+\sum_{\alpha}\pi_{G_\alpha}\left(\left[\prod_i X^{\alpha_i}\setminus\Delta\right]\cdot\prod_i A_i^{\alpha_i}\right)t^{|\alpha|} \tag{2.2.2}$$

endows $K_0(\mathrm{Var}_{\mathbb{C}})$ with a power structure. Equation (2.2.2) can be viewed as a motivic version of the combinatorial identity (2.2.1). Here $G_\alpha=\prod_i \mathfrak{S}_{\alpha_i}$ is the automorphism group of α, by $\Delta\subset\prod_i X^{\alpha_i}$ we mean the "big diagonal" (where at least two entries are equal), and we are viewing

$$\left[\prod_i X^{\alpha_i}\setminus\Delta\right]\cdot\prod_i A_i^{\alpha_i}\in \widetilde{K}_0^{G_\alpha}(\mathrm{Var}_{\mathbb{C}})$$

as an equivariant class, with G_α acting simultaneously on the two factors, so that it makes sense to apply the quotient map introduced in (2.1.4). Note that, if α has size k, the free quotient

$$\left(\prod_i X^{\alpha_i}\setminus\Delta\right)\Big/ G_\alpha$$

is canonically isomorphic to the stratum $\mathrm{Sym}^k_\alpha X\subset \mathrm{Sym}^k X$ parametrizing zero-cycles whose support is distributed according to α. The symmetric product plays a key role in the theory of power structures over motivic rings. The link is Theorem 2.2.2 below. Let

$$\zeta_{[X]}(t)=\sum_{n\geq 0}[\mathrm{Sym}^n X]t^n \tag{2.2.3}$$

be the Kapranov zeta function of the variety X.

THEOREM 2.2.2 ([36, Thm. 1]). *Equation* (2.2.2) *defines a power structure on* $K_0(\mathrm{Var}_{\mathbb{C}})$*, uniquely determined by the relation*

$$(1-t)^{-[X]}=\zeta_{[X]}(t).$$

Example 2.2.3. It is an immediate check that the Kapranov zeta function satisfies

$$\zeta_{[X]+[Y]} = \zeta_{[X]} \cdot \zeta_{[Y]}.$$

One has, for instance,

$$\zeta_{\mathbb{L}^n}(t) = \frac{1}{1-\mathbb{L}^n t}, \qquad \zeta_{[\mathbb{P}^n]}(t) = \prod_{i=0}^{n} \frac{1}{1-\mathbb{L}^i t}. \qquad \diamondsuit$$

It is often handy to rephrase motivic identities in terms of the *motivic exponential*, which is a group isomorphism[3]

$$\mathrm{Exp}: t\,K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!] \xrightarrow{\sim} 1 + t\,K_0(\mathsf{Var}_{\mathbb{C}})[\![t]\!]$$

defined by

$$\mathrm{Exp} \sum_{n>0} A_n t^n = \prod_{n>0} (1-t^n)^{-A_n}.$$

2.2.1 Geometric interpretation

The power structure of Theorem 2.2.2 has an insightful geometric interpretation, again due to Gusein-Zade, Luengo and Melle-Hernández [36]. It goes as follows. Let (A_n) be a sequence of algebraic varieties, and let X be another variety. Consider the series $A(t) = 1 + \sum_{n>0}[A_n]t^n$. If $[B_n]$ denotes the coefficient of t^n in $A(t)^{[X]}$ according to (2.2.2), then $[B_n]$ is in fact an effective class in $K_0(\mathsf{Var}_{\mathbb{C}})$, representing the algebraic variety

$$B_n = \coprod_{\alpha \vdash n} \left(\prod_i X^{\alpha_i} \setminus \Delta \times \prod_i A_i^{\alpha_i} \right) \Big/ G_\alpha$$

with G_α acting diagonally by permuting the factors. The points of B_n are in one to one correspondence with elements of the set

$$\left\{ (K,\phi) \;\middle|\; \begin{array}{c} K \subset X \text{ is a finite set, } \phi : K \to \coprod_{i>0} A_i \\ \text{is a map such that } \sum_{x \in K} \tau(\phi(x)) = n \end{array} \right\}, \tag{2.2.4}$$

where $\tau : \coprod_{i>0} A_i \to \mathbb{Z}$ is the map sending the whole A_i to the integer i.

2.2.2 Extensions

The zeta function satisfies

$$\zeta_{\mathbb{L}^s[X]} = \zeta_{[X]}(\mathbb{L}^s t)$$

for all $s \geq 0$. This determines a unique extension of the power structure on $K_0(\mathsf{Var}_{\mathbb{C}})$ to the localization $K_0(\mathsf{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. See [7, Section 2] for a further extension to $\mathcal{M}_{\mathbb{C}}$. There is also an extension to $K_0(\mathsf{St}_{\mathbb{C}})$, defined by

$$(1-t)^{-\mathbb{L}^s[X]} = (1-\mathbb{L}^s t)^{-[X]},$$

3 The group structures are the additive one on the source and the multiplicative one on the target.

where X is a variety and $s \in \mathbb{Z}$, see [38]. Regarding the geometric interpretation, it is not true anymore that the power structure on $K_0(\mathrm{St}_{\mathbb{C}})$ is effective: the coefficients of $A(t)^{[X]}$ may not represent any algebraic stack (with affine stabilizers) when A_i and X are stacks. However, they do represent an algebraic stack if $[X]$ is the class of a variety [19, Lemma 5]. In this case, the geometric interpretation (2.2.4) is still valid. The motivic exponential extends naturally to $\mathcal{M}_{\mathbb{C}}$ and to $K_0(\mathrm{St}_{\mathbb{C}})$ along with the power structure.

2.2.3 Examples

We now describe some applications of the power structure, in the context of the Hilbert scheme of points of a variety, and the stack of coherent sheaves of finite length on $\mathbb{A}^2$.

Let Y be a smooth quasi-projective variety of dimension d. Exploiting the geometric interpretation of the power structure, one can prove

$$\sum_{n\geq 0}[\mathrm{Hilb}^n Y]t^n = \left(\sum_{n\geq 0}[\mathrm{Hilb}^n(\mathbb{A}^d)_0]t^n\right)^{[Y]} \in K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!],$$

where $\mathrm{Hilb}^n(\mathbb{A}^d)_0$ is the punctual Hilbert scheme [37, Thm. 1]. Interpreting $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ as a homomorphism of power structures, one deduces from the above identity the numerical relation

$$\sum_{n\geq 0}\chi(\mathrm{Hilb}^n Y)t^n = \left(\sum_{n\geq 0}P_{d-1}(n)t^n\right)^{\chi(Y)}$$

where $P_{d-1}(n)$ is the number of $(d-1)$-dimensional partitions of n. If $d \leq 3$, there are well-known product formulas for these series, namely

$$\sum_{n\geq 0}\chi(\mathrm{Hilb}^n Y)t^n = \begin{cases} (1-t)^{-\chi(Y)} & \text{if } d=1 \\ \displaystyle\prod_{m\geq 1}(1-t^m)^{-\chi(Y)} & \text{if } d=2 \\ \displaystyle\prod_{m\geq 1}(1-t^m)^{-m\chi(Y)} & \text{if } d=3. \end{cases}$$

The case $d=1$ goes back to MacDonald, whereas the formulas for surfaces and threefolds have been proved by Göttsche and Cheah, respectively. No product formula is known for $d>3$. The corresponding motivic refinements for $d=1,2$ are given by

$$\sum_{n\geq 0}[\mathrm{Hilb}^n Y]t^n = \begin{cases} (1-t)^{-[Y]} & \text{if } d=1 \\ \displaystyle\prod_{m\geq 1}(1-\mathbb{L}^{m-1}t^m)^{-[Y]} & \text{if } d=2. \end{cases}$$

The motive of the Hilbert scheme of points on a smooth quasi-projective threefold is not that well-behaved, as the Hilbert scheme is singular. However, it is "virtually smooth", and the *virtual* motive $[\mathrm{Hilb}^n Y]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ defined in [7] is a nicer object. For a smooth quasi-projective threefold Y, we use the same notation as in [7] to denote the generating functions

$$\mathsf{Z}_Y(t) = \sum_{n\geq 0} \big[\mathrm{Hilb}^n Y\big]_{\mathrm{vir}} t^n, \qquad \mathsf{Z}_{\mathbb{A}^3,0}(t) = \sum_{n\geq 0} \big[\mathrm{Hilb}^n(\mathbb{A}^3)_0\big]_{\mathrm{vir}} t^n. \tag{2.2.5}$$

The virtual motive of the punctual Hilbert scheme $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ is defined in [7, Section 3]. We will later exploit the following result.

THEOREM 2.2.4 ([7, Prop. 4.2]). *Let Y be a smooth quasi-projective threefold. In $\mathcal{M}_{\mathbb{C}}[\![t]\!]$ one has the identity*

$$\mathsf{Z}_Y(t) = \mathsf{Z}_{\mathbb{A}^3,0}(t)^{[Y]}.$$

Another application of the power structure involves the stack $\mathsf{Coh}_n(\mathbb{A}^2)$ of coherent sheaves of length n on $\mathbb{A}^2$. One has

$$\mathsf{C}(t) = \left(\sum_{n\geq 0} \big[\mathsf{Coh}_n(\mathbb{A}^2)_0\big]\right)^{\mathbb{L}^2} = \mathrm{Exp}\left(\frac{\mathbb{L}^2}{\mathbb{L}-1}\frac{t}{1-t}\right), \tag{2.2.6}$$

where $\mathsf{Coh}_n(\mathbb{A}^2)_0 \subset \mathsf{Coh}_n(\mathbb{A}^2)$ is the closed substack of coherent sheaves entirely supported at the origin.

2.2.4 Punctual motives for $\mathbb{A}^2$

Let us focus on the affine surface $Y = \mathbb{A}^2$. Using the power structure, it is possible to extract from the formulas of the previous section the motivic contributions of the "punctual" motives, namely $[\mathrm{Hilb}^n(\mathbb{A}^2)_0]$ and $[\mathsf{Coh}_n(\mathbb{A}^2)_0]$. Knowing the first few coefficients of the corresponding generating series will help us providing evidence for a conjecture in Chapter 8.

For the Hilbert scheme, we get

$$\sum_{n\geq 0} [\mathrm{Hilb}^n(\mathbb{A}^2)_0] t^n = \prod_{m\geq 1} (1-\mathbb{L}^{m-1}t^m)^{-1}, \tag{2.2.7}$$

whose first terms are

$$\begin{aligned} 1 &+ t + (1+\mathbb{L})t^2 + (1+\mathbb{L}+\mathbb{L}^2)t^3 \\ &+ (1+\mathbb{L}+2\mathbb{L}^2+\mathbb{L}^3)t^4 + (1+\mathbb{L}+2\mathbb{L}^2+2\mathbb{L}^3+\mathbb{L}^4)t^5 + \cdots \end{aligned}$$

Remark 2.2.5. The n-th coefficient of the above series always contains a summand of the form $(\mathbb{L}+1)\mathbb{L}^{n-2}$. This motive is the class of the curvilinear locus, an open subscheme $C_n^0 \subset \mathrm{Hilb}^n(\mathbb{A}^2)_0$ that Briançon proved to be *dense* [14, Théorème V.3.2] and fibred over $\mathbb{P}^1 = \mathbb{P}(\mathfrak{m}/\mathfrak{m}^2)$ (the space of double points at the origin $0 \in \mathbb{A}^2$), with fibre $\mathbb{A}^{n-2}$ [14, Prop. IV.1.1]. Here $\mathfrak{m} = (x, y)$ is the ideal of the origin. The remaining class is the class of its complement. For instance, if $n = 3$, the complement has class equal to 1, corresponding to the single non-curvilinear ideal $\mathfrak{m}^2 \subset \mathbb{C}[x, y]$. For $n = 4$, the complement has class $1+\mathbb{L}+\mathbb{L}^2$. $\diamondsuit$

For the stack of coherent sheaves, one can use the Feit–Fine formula (Theorem 2.1.4) to compute

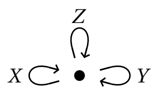
$$\sum_{n\geq 0}[\mathrm{Coh}_n(\mathbb{A}^2)_0]t^n = \prod_{k\geq 1}\prod_{m\geq 1}(1-\mathbb{L}^{-k}t^m)^{-1}. \tag{2.2.8}$$

The first few terms are

$$1+\frac{1}{\mathbb{L}-1}t+\left(\frac{1}{\mathrm{GL}_2}+\frac{\mathbb{L}+1}{\mathbb{L}(\mathbb{L}-1)}\right)t^2+\cdots$$

2.3 Virtual motives of the 3-loop quiver

Let $n\geq 0$ and $p\geq 1$ be integers, and let V be an n-dimensional complex vector space. The affine space $\mathrm{End}(V)^3$ parametrizes n-dimensional representations of the *three loop quiver*, namely the quiver



consisting of one node and three loops. We write L_3 for this quiver. We have $\mathrm{GL}_n = \mathrm{GL}(V)$ acting on $\mathrm{Rep}(\mathsf{L}_3) = \mathrm{End}(V)^3$ by simultaneous conjugation. The quotient stack $[\mathrm{End}(V)^3/\mathrm{GL}_n]$ parametrizes isomorphism classes of representations of L_3. Instead of studying this stack, we work with *framed* representations: as a warm-up for the computations we will be doing in Section 7.2, we study here the motivic DT invariants of the three loop quiver (associated to a certain super-potential). We follow closely the computation of [7, Theorem 3.7] where in the case $p=1$ the authors found the product formula

$$\mathsf{Z}_{\mathbb{A}^3}(t)=\prod_{m\geq 1}\prod_{k=0}^{m-1}\left(1-\mathbb{L}^{2+k-m/2}t^m\right)^{-1}.$$

The series $\mathsf{Z}_{\mathbb{A}^3}$, recalled in (2.2.5), is the motivic DT partition function of the Hilbert scheme, representing the natural refinement of the zero-dimensional DT theory of $\mathbb{A}^3$, given by

$$\sum_{n\geq 0}\chi_{\mathrm{vir}}(\mathrm{Hilb}^n\mathbb{A}^3)t^n=\prod_{m\geq 1}(1-(-t)^m)^{-m}=M(-t),$$

where $M(t)$ is the MacMahon function. We stress that nothing is original in this section. However, it is a good opportunity to fix some notation and anticipate the strategy used in later computations. Also, in Section 7.2 we will need a relation we will soon get along the way, namely (2.3.7) below.

2.3.1 Critical loci attached to the quiver

Let us form the affine space

$$\mathcal{R}(n,p)=\mathrm{End}(V)^3\times V^p.$$

Definition 2.3.1. For a point $x = (A, B, C, v_1, \dots, v_p) \in \mathcal{R}(n, p)$, the vector space

$$\mathrm{Span}(x) = \mathrm{Span}_{\mathbb{C}} \left\{ A^{\ell_1} B^{\ell_2} C^{\ell_3} \cdot v_i \,\middle|\, \ell_j \geq 0, 1 \leq i \leq p \right\} \subset V$$

will be called the *span* of x. ♢

Definition 2.3.2. Let $U_n^p \subset \mathcal{R}(n, p)$ be the open subscheme consisting of points x whose span is exactly V. When $p = 2$, we shall simply write U_n and $\mathcal{R}_n$ instead of U_n^2 and $\mathcal{R}(n, 2)$. ♢

THEOREM 2.3.3 ([43]). *The open set U_n^p coincides with the set of semistable points for the* GL_n*-action on* $\mathcal{R}(n, p)$ *given by*

$$g \cdot (A, B, C, v_1, \dots, v_p) = (A^g, B^g, C^g, g v_1, \dots, g v_p),$$

and linearized by the character $\det : \mathrm{GL}_n \to \mathbb{G}_m$.

LEMMA 2.3.4. *Points in U_n^p have trivial stabilizer.*

PROOF. If $g \in \mathrm{GL}_n$ fixes $(A, B, C, v_1, \dots, v_p)$, then each v_i lies in the invariant subspace $\ker(g - \mathrm{id}) \subset V$. But by definition of U_n^p, the smallest invariant subspace containing $v_1, \dots, v_p$ is V itself, hence $g = \mathrm{id}$. □

The lemma implies that there is no difference between stable and semistable. Stability for framed representations can be thought of as a limit of King stability. Theorem 2.3.3 allows one to construct the (smooth and quasi-projective) geometric quotient

$$U_n^p / \mathrm{GL}_n = \mathcal{R}(n, p) /\!\!/_{\det} \mathrm{GL}_n,$$

which is the moduli space of p-framed n-dimensional representations of L_3. When $p = 1$, this space is also known as the *non-commutative Hilbert scheme*, sometimes denoted

$$\mathrm{NCHilb}_3^n = U_n^1 / \mathrm{GL}_n. \tag{2.3.1}$$

For a general quiver Q, let $\mathbb{C}Q$ denote the path algebra of Q. An element of the quotient

$$\mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$$

is called a *super-potential* if it is represented by a (finite) sum of loops. For the three loop quiver, we have

$$\mathbb{C}\mathsf{L}_3 = \mathbb{C}\langle X, Y, Z \rangle$$

and we look at the super-potential

$$W = X(YZ - ZY) \in \mathbb{C}\mathsf{L}_3 / [\mathbb{C}\mathsf{L}_3, \mathbb{C}\mathsf{L}_3] = \mathbb{C}[X, Y, Z],$$

viewed as a combination of cycles uniquely defined up to cyclic permutations. Then W induces a regular map $\widetilde{\mathrm{W}}_n : \mathcal{R}(n, p) \to \mathbb{A}^1$ defined by

$$\widetilde{\mathrm{W}}_n(A, B, C, v_1, \dots, v_p) = \mathrm{Tr}\, A[B, C].\text{[4]} \tag{2.3.2}$$

4 As p is fixed, it is omitted from the notation regarding the maps.

Note that the map does not interact with the vectors. We let W_n be the restriction of (2.3.2) to U_n^p and we observe that it descends to the quotient U_n^p/GL_n, since it is GL_n-invariant. This defines a regular map

$$\mathsf{w}_n : U_n^p/\mathrm{GL}_n \to \mathbb{A}^1.$$

We are interested in the canonical virtual motive attached to the critical locus

$$D_{n,p} = Z(d\mathsf{w}_n) \subset U_n^p/\mathrm{GL}_n.$$

Example 2.3.5. It is the content of [7, Prop. 3.1] that

$$D_{n,1} = \mathrm{Hilb}^n(\mathbb{A}^3) \subset \mathrm{NCHilb}_3^n. \qquad \diamondsuit$$

Aside 2.3.1. One can work with more than three matrices and obtain a (smooth) scheme NCHilb_d^n for all d (using again just one cyclic vector). This is tightly related to representations of the free algebra $\mathbb{C}\langle x_1,\dots,x_d\rangle$. However, only in dimension $d=3$ one can explicitly describe the (commutative) Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^3)$ as the critical locus of a function $\mathrm{NCHilb}_3^n \to \mathbb{A}^1$. No such thing seems to be possible for $d>3$.

One may also forget about GIT and construct noncommutative Hilbert schemes via a functorial approach: one ends up with moduli schemes of *left ideals* of codimension n in very general algebras (and such schemes are well-known to be smooth when the algebra is formally smooth), see for instance [46, 58, 80, 29]. When the algebra R one starts with is commutative, this construction yields the (commutative) Hilbert scheme $\mathrm{Hilb}^n(\operatorname{Spec} R)$. We will touch upon this functorial point of view in Section 6.2.1.

2.3.2 Computing the partition function

We now derive a product formula for the motivic generating series

$$\mathsf{F}_p(t) = \sum_{n\geq 0} [D_{n,p}]_{\mathrm{vir}}\, t^n \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}[\![t]\!]. \tag{2.3.3}$$

In fact, the coefficients of this series live in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. To see this, consider the action of the torus $\mathbf{T} = \mathbb{G}_m^3$ on U_n^p by

$$t\cdot(A,B,C,v_1,\dots,v_p) = (t_1A, t_2B, t_3C, t_1t_2t_3v_1,\dots,t_1t_2t_3v_p), \tag{2.3.4}$$

along with the primitive character $\chi(t) = t_1t_2t_3$. Then this action descends to an action on U_n^p/GL_n and both W_n and w_n are $\mathbf{T}$-equivariant with respect to χ. Moreover, the induced actions of the diagonal subtorus $\mathbb{G}_m \subset \mathbf{T}$ are circle compact, as in the proof of [7, Lemma 3.4]. Then Theorem 2.1.16 ensures that

$$[\phi_{\mathsf{W}_n}] = [\mathsf{W}_n^{-1}(1)] - [\mathsf{W}_n^{-1}(0)] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}},$$

and similarly for $[\phi_{\mathsf{w}_n}]$. Since $\dim U_n^p/\mathrm{GL}_n = 2n^2+pn$, in $\mathcal{M}_{\mathbb{C}}$ one has

$$[D_{n,p}]_{\mathrm{vir}} = -\mathbb{L}^{-n^2-pn/2}[\phi_{\mathsf{w}_n}],$$

with

$$[\phi_{\mathsf{w}_n}] = \frac{[\phi_{\mathsf{W}_n}]}{\mathrm{GL}_n} = \frac{[\phi_{\mathsf{W}_n}]}{\mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!} \in \mathcal{M}_{\mathbb{C}}\left[(1-\mathbb{L}^i)^{-1} \,\middle|\, i\geq 1\right]. \tag{2.3.5}$$

So we need to compute the absolute motivic vanishing cycle $[\phi_{\mathsf{W}_n}]$.

PROPOSITION 2.3.6. *The series* (2.3.3) *is given by*

$$\mathsf{F}_p(t) = \prod_{m\geq 1}\prod_{k=0}^{pm-1}(1-\mathbb{L}^{2+k-pm/2}t^m)^{-1} \in \mathcal{M}_{\mathbb{C}}[\![t]\!].$$

First of all, let us identify $\mathcal{R}(n,p)$ with affine space $\mathbb{A}^{3n^2+pn}$. Write

$$Y_n = \widetilde{\mathsf{W}}_n^{-1}(0) \subset \mathcal{R}(n,p), \qquad Z_n = \widetilde{\mathsf{W}}_n^{-1}(1) \subset \mathcal{R}(n,p)$$

for the special and the generic fibre of $\widetilde{\mathsf{W}}_n$. Since $\widetilde{\mathsf{W}}_n$ is $\mathbf{T}$-equivariant with respect to χ via (2.3.4), by (2.1.6) we have an isomorphism $\mathbb{G}_m \times Z_n \cong \mathcal{R}(n,p)\setminus Y_n$, whence the motivic relation

$$[Y_n]+(\mathbb{L}-1)[Z_n] = \mathbb{L}^{3n^2+pn}.$$

Setting

$$\omega_n = [Y_n]-[Z_n],$$

we can rewrite the above equality as

$$(1-\mathbb{L})\omega_n = \mathbb{L}^{3n^2+pn}-\mathbb{L}[Y_n]. \tag{2.3.6}$$

Now, Y_n decomposes as $Y_n' \amalg Y_n''$, where Y_n' consists of those tuples in Y_n satisfying $[B,C]=0$. Then $Y_n' \cong \mathbb{A}^{n^2+pn}\times C_n$, while the complement Y_n'' is a hyperplane bundle over $\mathbb{A}^{2n^2}\setminus C_n$. Hence

$$[Y_n] = [Y_n']+[Y_n''] = \mathbb{L}^{n^2+pn}[C_n]+(\mathbb{L}^{2n^2}-[C_n])\mathbb{L}^{n^2-1+pn}.$$

This yields, substituting in (2.3.6), the identity

$$\begin{aligned}(1-\mathbb{L})\omega_n &= \mathbb{L}^{3n^2+pn}-\mathbb{L}^{n^2+pn+1}[C_n]-(\mathbb{L}^{2n^2}-[C_n])\mathbb{L}^{n^2+pn}\\ &= \mathbb{L}^{3n^2+pn}-\mathbb{L}^{n^2+pn+1}[C_n]-\mathbb{L}^{3n^2+pn}+\mathbb{L}^{n^2+pn}[C_n]\\ &= (1-\mathbb{L})\mathbb{L}^{n^2+pn}[C_n],\end{aligned}$$

from which we get

$$\omega_n = \mathbb{L}^{n(n+p)}[C_n]. \tag{2.3.7}$$

Let us now define, for $0\leq k\leq n$, the subset

$$X^k = \left\{ x\in\mathcal{R}(n,p) \,\middle|\, \text{the span of } x \text{ is } k\text{-dimensional}\right\} \subset \mathcal{R}(n,p).$$

Then, setting $Y_n^k = Y_n \cap X^k$ and $Z_n^k = Z_n\cap X^k$, we find that

$$Y_n^n = \mathsf{W}_n^{-1}(0), \qquad Z_n^n = \mathsf{W}_n^{-1}(1).$$

Defining

$$\omega_n^k = [Y_n^k]-[Z_n^k],$$

we see that, because of (2.3.5), the motivic difference we are interested in is

$$[\phi_{\mathsf{W}_n}] = -[Y_n^n]+[Z_n^n] = -\omega_n^n. \tag{2.3.8}$$

We can then write

$$\begin{aligned}[D_{n,p}]_{\mathrm{vir}} &= -\mathbb{L}^{-n^2-pn/2}[\phi_{\mathsf{W}_n}]\\ &= -\mathbb{L}^{-n^2-pn/2}\frac{-\omega_n^n}{\mathsf{GL}_n} = \frac{\omega_n^n}{\mathbb{L}^{\frac{3n^2+n(p-1)}{2}}[n]_{\mathbb{L}}!}.\end{aligned} \tag{2.3.9}$$

Computing $[Y_n^k]$

The map $h : Y_n^k \to \mathrm{Gr}(k,V)$ sending a point to its span is a Zariski locally trivial fibration. Let us compute the motive of the fibre. For a given $\Lambda \in \mathrm{Gr}(k,V)$, we can choose a basis of V so that the first k vectors of the basis belong to Λ. Then, any $P = (A,B,C,v_1,\dots,v_p) \in h^{-1}(\Lambda)$ will be of the following form:

$$A = \begin{pmatrix} A_0 & A' \\ 0 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} B_0 & B' \\ 0 & B_1 \end{pmatrix}, \qquad C = \begin{pmatrix} C_0 & C' \\ 0 & C_1 \end{pmatrix}, \qquad v_i = \begin{pmatrix} v_{i0} \\ 0 \end{pmatrix},$$

where A_0, B_0, C_0 are $k \times k$ matrices, A_1, B_1, C_1 are $(n-k)\times(n-k)$ matrices, A', B', C' are $k \times (n-k)$ matrices, and finally v_{i0} are k-vectors, which for convenience we collect together in the compact notation $\mathbf{v} = (v_{10},\dots,v_{p0})$. We certainly have

$$\mathrm{Tr}\,A[B,C] = \mathrm{Tr}\,A_0[B_0,C_0] + \mathrm{Tr}\,A_1[B_1,C_1],$$

and if we set, for shorthand, $\mathrm{Tr}_i = \mathrm{Tr}\,A_i[B_i,C_i]$, we get

$$\begin{aligned} h^{-1}(\Lambda) &= \left\{ (A_0,B_0,C_0,\mathbf{v},A_1,B_1,C_1,A',B',C') \,\middle|\, \mathrm{Tr}_0 + \mathrm{Tr}_1 = 0 \right\} \\ &= \mathbb{A}^{3k(n-k)} \times (S \amalg T), \end{aligned}$$

where $\mathbb{A}^{3k(n-k)}$ takes care of A', B', C' and

$$\begin{aligned} S &= \left\{ (A_0,B_0,C_0,\mathbf{v},A_1,B_1,C_1) \,\middle|\, \mathrm{Tr}_0 = \mathrm{Tr}_1 = 0 \right\}, \\ T &= \left\{ (A_0,B_0,C_0,\mathbf{v},A_1,B_1,C_1) \,\middle|\, \mathrm{Tr}_0 = -\mathrm{Tr}_1 \neq 0 \right\}. \end{aligned}$$

There are isomorphisms

$$\begin{aligned} \psi_S &: S \times \mathbb{A}^{p(n-k)} \xrightarrow{\sim} Y_k^k \times Y_{n-k} \\ \psi_T &: T \times \mathbb{A}^{p(n-k)} \xrightarrow{\sim} \mathbb{C}^\times \times Z_k^k \times Z_{n-k} \end{aligned}$$

defined as follows.

- If $\mathbf{e} = (e_1,\dots,e_p) \in \mathbb{A}^{p(n-k)}$ is a p-tuple of $(n-k)$-vectors, ψ_S sends

$$(A_0,B_0,C_0,\mathbf{v},A_1,B_1,C_1;\mathbf{e}) \mapsto (A_0,B_0,C_0,\mathbf{v};A_1B_1,C_1,\mathbf{e}).$$

- Similarly, ψ_T is defined by

$$(A_0,B_0,C_0,\mathbf{v},A_1,B_1,C_1;\mathbf{e}) \mapsto (\mathrm{Tr}_0;\mathrm{Tr}_0^{-1}A_0,B_0,C_0,\mathbf{v};\mathrm{Tr}_1^{-1}A_1,B_1,C_1,\mathbf{e}).$$

Hence we find

$$\begin{aligned} [Y_n^k] &= [\mathrm{Gr}(k,V)]\mathbb{L}^{3k(n-k)}([S]+[T]) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-p)(n-k)}([Y_k^k][Y_{n-k}] + (\mathbb{L}-1)[Z_k^k][Z_{n-k}]). \end{aligned}$$

Computing $[Z_n^k]$

We compute the fibre of the Zariski fibration $l : Z_n^k \to \mathrm{Gr}(k,V)$. In this case, the matrices A', B', C' still play no role, thus the fiber decomposes as

$$l^{-1}(\Lambda) = \mathbb{A}^{3k(n-k)} \times (S_1 \amalg S_2 \amalg S_3)$$

where:

$$\begin{aligned}
S_1 &= \left\{ (A_0, B_0, C_0, \mathbf{v}, A_1, B_1, C_1) \,\middle|\, \mathrm{Tr}_0 = 0, \mathrm{Tr}_1 = 1 \right\},\\
S_2 &= \left\{ (A_0, B_0, C_0, \mathbf{v}, A_1, B_1, C_1) \,\middle|\, \mathrm{Tr}_0 = 1, \mathrm{Tr}_1 = 0 \right\},\\
S_3 &= \left\{ (A_0, B_0, C_0, \mathbf{v}, A_1, B_1, C_1) \,\middle|\, \mathrm{Tr}_0 = 1 - \mathrm{Tr}_1 \neq 0, 1 \right\}.
\end{aligned}$$

As before, there are isomorphisms

$$\begin{aligned}
S_1 \times \mathbb{A}^{p(n-k)} &\xrightarrow{\sim} Y_k^k \times Z_{n-k}\\
S_2 \times \mathbb{A}^{p(n-k)} &\xrightarrow{\sim} Z_k^k \times Y_{n-k}\\
S_3 \times \mathbb{A}^{p(n-k)} &\xrightarrow{\sim} (\mathbb{C}^\times \setminus \{1\}) \times Z_k^k \times Z_{n-k}.
\end{aligned}$$

Hence we find:

$$\begin{aligned}
[Z_n^k] &= [\mathrm{Gr}(k,V)]\mathbb{L}^{3k(n-k)}([S_1] + [S_2] + [S_3])\\
&= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-p)(n-k)}([Y_k^k][Z_{n-k}] + [Z_k^k][Y_{n-k}] + (\mathbb{L}-2)[Z_k^k][Z_{n-k}]).
\end{aligned}$$

The key recursion

We can now write the motive ω_n^k as follows:

$$\begin{aligned}
\omega_n^k &= [Y_n^k] - [Z_n^k]\\
&= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-p)(n-k)}([Y_k^k]\omega_{n-k} - [Z_k^k]\omega_{n-k})\\
&= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-p)(n-k)}\omega_{n-k}\omega_k^k\\
&= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-p)(n-k)}\mathbb{L}^{(n-k)^2+p(n-k)}[C_{n-k}]\omega_k^k\\
&= [\mathrm{Gr}(k,V)]\mathbb{L}^{(n-k)(n+2k)}[C_{n-k}]\omega_k^k.
\end{aligned}$$

Since $Y_n = \coprod_k Y_n^k$ and $Z_n = \coprod_k Z_n^k$, we find

$$\begin{aligned}
\omega_n^n &= \omega_n - \sum_{k=0}^{n-1} \omega_n^k\\
&= \mathbb{L}^{n^2+pn}[C_n] - \sum_{k=0}^{n-1} [\mathrm{Gr}(k,V)]\mathbb{L}^{(n-k)(n+2k)}[C_{n-k}]\omega_k^k.
\end{aligned} \tag{2.3.10}$$

We know by (2.3.9) that

$$[D_{n,p}]_{\mathrm{vir}} = \frac{\omega_n^n}{\mathbb{L}^{\frac{3n^2+n(p-1)}{2}}[n]_{\mathbb{L}}!},$$

so we can divide (2.3.10) by $\mathbb{L}^{\frac{3n^2+n(p-1)}{2}}[n]_{\mathbb{L}}!$ and rearrange to get

$$\widetilde{c}_n \mathbb{L}^{pn/2} = \sum_{k=0}^{n} \widetilde{c}_{n-k} \mathbb{L}^{-(n-k)p/2} \cdot [D_{k,p}]_{\mathrm{vir}}.$$

We have used the expression (2.1.1) for the motive of the Grassmannian, along with the class

$$\widetilde{c}_n = \frac{[C_n]}{\mathsf{GL}_n} = \frac{[C_n]}{\mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!}$$

defined in (2.1.3). Multiplying by t^n and summing, we get

$$\sum_{n\geq 0} \widetilde{c}_n (t\mathbb{L}^{p/2})^n = \mathsf{F}_p(t) \cdot \sum_{n\geq 0} \widetilde{c}_n (t\mathbb{L}^{-p/2})^n.$$

Using Theorem 2.1.4 we find

$$\begin{aligned}
\mathsf{F}_p(t) &= \frac{\mathsf{C}(t\mathbb{L}^{p/2})}{\mathsf{C}(t\mathbb{L}^{-p/2})} \\
&= \prod_{m\geq 1}\prod_{j\geq 0} \frac{(1-\mathbb{L}^{1-j+pm/2}t^m)^{-1}}{(1-\mathbb{L}^{1-j-pm/2}t^m)^{-1}} \\
&= \prod_{m\geq 1}\prod_{j=0}^{pm-1} (1-\mathbb{L}^{1-j+pm/2}t^m)^{-1} \\
&= \prod_{m\geq 1}\prod_{k=0}^{pm-1} (1-\mathbb{L}^{2+k-pm/2}t^m)^{-1}.
\end{aligned} \tag{2.3.11}$$

The proof of Proposition 2.3.6 is complete.

Part II

ENUMERATIVE DT INVARIANTS

3 | THE KUMMER SCHEME OF AN ABELIAN THREEFOLD

This section is joint work with M. Gulbrandsen [35].

3.1 Introduction

Let $n > 0$ be an integer. The n-th *generalized Kummer scheme* $K^n X$ of an abelian variety X is the fibre over 0_X of the composite map

$$\operatorname{Hilb}^n X \to \operatorname{Sym}^n X \to X,$$

where the first arrow is the Hilbert–Chow morphism and the second arrow takes a cycle to the weighted sum of its supporting points. The purpose of this note is to prove the following formula, which is the three-dimensional case of a conjecture from [34]:

THEOREM 3.1.1. *Let X be an abelian threefold. The Euler characteristic of its generalized Kummer Scheme $K^n X$ is*

$$\chi(K^n X) = n^5 \sum_{d|n} d^2.$$

Simultaneously with and independent of our work, Shen [72] has proven the conjecture in [34] for X an abelian variety of arbitrary dimension g, stating that

$$\sum_{n\geq 0} P_{g-1}(n) q^n = \exp\left(\sum_{n\geq 1} \frac{\chi(K^n X)}{n^{2g}} q^n\right), \tag{3.1.1}$$

where $P_d(n)$ denotes the number of d-dimensional partitions of n. In fact, Shen proves a further generalization of this to the case of a product $X \times Y$, where one factor X is an abelian variety, and the other factor Y is an arbitrary quasi-projective variety. For $g = 3$, the formula in Theorem 3.1.1 is recovered from (3.1.1) by applying MacMahon's product formula for plane partitions, cf. [74, Cor. 7.20.3].

One motivation for the computation of $\chi(K^n X)$ is as a test case for Donaldson–Thomas invariants for abelian threefolds, as developed in [34]. In particular (see *loc. cit.*), the Donaldson–Thomas invariant of the moduli stack $[K^n X / X_n]$ is the rational number

$$\frac{(-1)^{n+1}}{n^6} \chi(K^n X) = \frac{(-1)^{n+1}}{n} \sum_{d|n} d^2.$$

The formula (3.1.1) could be motivated by formally expanding Cheah's formula for the Euler characteristic of Hilbert schemes of points (see [22], and also [37] for a motivic refinement), up to first order in $\chi(X)$, as follows:

$$
\begin{array}{c}
1+\sum_{n\geq 1}\chi(\mathrm{Hilb}^n X)q^n = 1+\chi(X)\sum_{n\geq 1}\frac{\chi(K^nX)}{n^{2g}}q^n \\
\| \\
\exp\left(\chi(X)\log\sum_{n\geq 0}P_{g-1}(n)q^n\right) = 1+\chi(X)\log\sum_{n\geq 0}P_{g-1}(n)q^n.
\end{array}
$$

The top equality comes from the étale cover $X\times K^nX \to \mathrm{Hilb}^n X$ of degree n^6, given by the translation action of X on the Hilbert scheme. The vertical equality is Cheah's formula. For the bottom equality, we treat $\chi(X)^2$ as zero when expanding exp.

Conventions. We work over $\mathbb{C}$. The symbol χ denotes the topological Euler characteristic. We denote by $\alpha \vdash n$ (one-dimensional) partitions of $n=\sum_i i\alpha_i$, corresponding to classical Young tableaux. The number of d-dimensional partitions of n is denoted $P_d(n)$. A higher dimensional partition can be seen as a generalized Young tableau, with $(d+1)$-dimensional boxes taking the role of squares. The convention is to set $P_d(0)=1$.

3.2 Proof of the conjecture

3.2.1 Stratification

The Hilbert scheme of points of any quasi-projective variety X admits a natural stratification by partitions,

$$\mathrm{Hilb}^n X = \coprod_{\alpha\vdash n}\mathrm{Hilb}^n_\alpha X$$

where $\mathrm{Hilb}^n_\alpha X$ denotes the (locally closed) locus of subschemes of X having exactly α_i components of length i. Let X be an abelian variety. Letting $K^n_\alpha X = K^nX\cap \mathrm{Hilb}^n_\alpha X$, we get an induced stratification of the Kummer scheme,

$$K^nX = \coprod_{\alpha\vdash n} K^n_\alpha X. \tag{3.2.1}$$

For each partition $\alpha\vdash n$, let us define the subscheme

$$V_\alpha = \left\{\xi\in \mathrm{Sym}^n_\alpha X \,\middle|\, \Sigma\xi = 0\right\}\subset \mathrm{Sym}^n_\alpha X$$

where Σ denotes addition of zero cycles under the group law on X. The Hilbert–Chow morphism $\mathrm{Hilb}^n X\to \mathrm{Sym}^n X$ restricts to morphisms

$$\pi_\alpha : K^n_\alpha X\to V_\alpha.$$

Fixing a point in V_α amounts to fixing the supporting points of the corresponding cycle and their multiplicities. Thus, each fibre of π_α is isomorphic to a product of punctual Hilbert schemes:

$$F_\alpha \cong \prod_i \mathrm{Hilb}^i(\mathbb{A}^3)_0^{\alpha_i}.$$

Hence, using (3.2.1), we find

$$\chi(K^n X) = \sum_{\alpha \vdash n} \chi(V_\alpha) \prod_i P_2(i)^{\alpha_i}, \tag{3.2.2}$$

where we have used $P_{d-1}(n) = \chi(\mathrm{Hilb}^n(\mathbb{A}^d)_0)$ (see [27] for $d = 2$ and [22, 37] for the general case).

Strategy of proof

Let $\sigma_2(n) = \sum_{d|n} d^2$ denote the square sum of divisors of an integer n. As is well known [1], these are related to the number of plane partitions by

$$nP_2(n) = \sum_{k=1}^{n} \sigma_2(k) P_2(n-k). \tag{3.2.3}$$

Let us define, for $\alpha \vdash n$, integers $c(\alpha) \in \mathbb{Z}$ by the recursion

$$c(\alpha) = \begin{cases} n & \text{if } \alpha = (n^1), \\ -\sum_{i, \alpha_i \neq 0} c(\hat{\alpha}^i) & \text{otherwise.} \end{cases} \tag{3.2.4}$$

Here, for a partition $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots \ell^{\alpha_\ell}) \vdash n$, with $\alpha_i \neq 0$, we let

$$\hat{\alpha}^i = (1^{\alpha_1} \cdots i^{\alpha_i - 1} \cdots \ell^{\alpha_\ell}) \vdash n - i. \tag{3.2.5}$$

We shall prove Theorem 3.1.1 in two steps, given by the two Lemmas that follow.

LEMMA 3.2.1. *The square sum of divisors σ_2 can be expressed in terms of the number of plane partitions P_2 as follows:*

$$\sigma_2(n) = \sum_{\alpha \vdash n} c(\alpha) \prod_i P_2(i)^{\alpha_i}. \tag{3.2.6}$$

LEMMA 3.2.2. *The Euler characteristics $\chi(V_\alpha)/n^5$ equal the numbers $c(\alpha)$ defined by recursion* (3.2.4)*.*

Assuming the two Lemmas, the main theorem follows:

PROOF OF THEOREM 3.1.1. Equation (3.2.2) gives

$$\begin{aligned} \frac{\chi(K^n X)}{n^5} &= \sum_{\alpha \vdash n} \frac{\chi(V_\alpha)}{n^5} \prod_i P_2(i)^{\alpha_i} \\ &= \sum_{\alpha \vdash n} c(\alpha) \prod_i P_2(i)^{\alpha_i} \\ &= \sigma_2(n). \end{aligned}$$

We have applied Lemma 3.2.2 in the second equality, and Lemma 3.2.1 in the last equality. □

3.2.2 A recursion

Let us introduce the shorthand

$$f(\alpha) = \prod_i P_2(i)^{\alpha_i}.$$

Expand the right hand side of (3.2.6), using the definition of $c(\alpha)$, to get

$$\sum_{\alpha \vdash n} c(\alpha) f(\alpha) = nP_2(n) - \sum_{\substack{\alpha \vdash n \\ \alpha \neq (n^1)}} \sum_{\substack{j \geq 1 \\ \alpha_j \neq 0}} c(\hat{\alpha}^j) f(\hat{\alpha}^j). \tag{3.2.7}$$

On the other hand, by induction on n, the identity (3.2.3) gives

$$\begin{aligned} \sigma_2(n) &= nP_2(n) - \sum_{k=1}^{n-1} \sigma_2(k) P_2(n-k) \\ &= nP_2(n) - \sum_{k=1}^{n-1} \sum_{\beta \vdash k} c(\beta) f(\beta) P_2(n-k). \end{aligned} \tag{3.2.8}$$

The sets over which the double sums in (3.2.7) and (3.2.8) run are clearly identified via $(k, \beta) = (n-j, \hat{\alpha}^j)$. Since $f(\alpha) = P_2(j) f(\hat{\alpha}^j)$, it follows that the two expressions (3.2.7) and (3.2.8) are identical. Lemma 3.2.1 is established.

3.2.3 An incidence correspondence

In this section we prove Lemma 3.2.2. The technique used is very similar to the one adopted in [32].

Later on, we will need the following:

Remark 3.2.3. Let $\alpha = (n^1)$. Then V_α is in bijection with the subgroup $X_n \subset X$ of n-torsion points in X. This implies that $\chi(V_\alpha) = \chi(X_n) = n^6$. In other words, $\chi(V_\alpha)/n^5 = n = c(\alpha)$. $\diamondsuit$

Fix a partition $\alpha \vdash n$ different from (n^1), and an index i such that $\alpha_i \neq 0$. We will compute $\chi(V_\alpha)$ in terms of the partition $\hat{\alpha}^i \vdash n-i$, thanks to an incidence correspondence between the spaces $V_\alpha \subset \operatorname{Sym}^n_\alpha X$ and $V_{\hat{\alpha}^i} \subset \operatorname{Sym}^{n-i}_{\hat{\alpha}^i} X$.

Let us define the subscheme

$$I = \left\{ (a, b; \xi) \in X^2 \times V_\alpha \,\middle|\, \operatorname{mult}_a \xi = i,\ (n-i)b = ia \text{ in } X \right\} \subset X^2 \times V_\alpha.$$

We use the incidence correspondence

$$\begin{tikzcd} I \arrow[r, "\phi"] \arrow[d, "\psi"'] & V_\alpha \\ V_{\hat{\alpha}^i} & \end{tikzcd}$$

where the map ϕ is the one induced by the second projection, and ψ sends $(a, b; \xi)$ to the cycle $T_b(\xi - ia)$, where T_b is translation by $b \in X$.

The strategy is to compute $\chi(I)$ twice: by means of the fibres of ϕ and ψ respectively. This will enable us to compare $\chi(V_\alpha)$ and $\chi(V_{\hat{\alpha}^i})$.

Fibres of ϕ. Let $\xi \in V_\alpha$. This means $\xi \in \mathrm{Sym}^n_\alpha X$ and $\sum \xi = 0$ in X. We have

$$\phi^{-1}(\xi) = \left\{ (a,b) \in X^2 \,\middle|\, \mathrm{mult}_a\, \xi = i,\ (n-i)b = ia \right\} \subset X^2.$$

Let $a_1, \dots, a_{\alpha_i}$ be the α_i points, in the support of ξ, having multiplicity i (recall that i is fixed). Then

$$\phi^{-1}(\xi) = \coprod_{1 \le j \le \alpha_i} H_j,$$

where $H_j = \{ b \in X \mid (n-i)b = ia_j \}$. Each H_j is the kernel of the translated isogeny $b \mapsto (n-i)b - ia_j$, which has degree $(n-i)^6$, so $\chi(H_j) = (n-i)^6$. This yields $\chi(\phi^{-1}(\xi)) = \alpha_i (n-i)^6$. Hence,

(3.2.9) $$\chi(I) = \chi(V_\alpha)\alpha_i (n-i)^6.$$

Fibres of ψ. Let $C \in V_{\hat{\alpha}^i}$. A point $(a,b;\xi) \in \psi^{-1}(C)$ determines ξ as

$$\xi = T_b^{-1}(C) + ia,$$

and the condition $\mathrm{mult}_a\, \xi = i$ translates into $\mathrm{mult}_a (T_b^{-1}(C) + ia) = i$, which means $a \notin \mathrm{Supp}\,(T_b^{-1}(C))$, or $a + b \notin \mathrm{Supp}\,(C)$.

Let us define the subscheme

$$B = \left\{ (a,b) \,\middle|\, (n-i)b = ia \right\} \subset X^2.$$

Then we note that

$$\psi^{-1}(C) = \left\{ (a,b) \in B \,\middle|\, a + b \notin \mathrm{Supp}\,(C) \right\} = B \setminus \coprod_{c \in \mathrm{Supp}\,(C)} Y_c,$$

where

$$Y_c = \left\{ (a,b) \in B \,\middle|\, a + b = c \right\} \cong \{ b \in X \mid nb = ic \} \cong X_n.$$

Now, if we map $B \to X$ through the second projection, we see that the fibres are all isomorphic (to X_i, the group of i-torsion points in X). Hence, as $\chi(X) = 0$, we find that $\chi(B) = 0$. Thus, remembering that $\mathrm{Supp}\,(C)$ consists of $(\sum_i \alpha_i) - 1$ distinct points, we find

$$\chi(\psi^{-1}(C)) = -\sum_{c \in \mathrm{Supp}\,(C)} \chi(Y_c) = -n^6 \cdot \left(\sum_i \alpha_i - 1 \right).$$

Finally,

(3.2.10) $$\chi(I) = -\chi(V_{\hat{\alpha}^i}) n^6 \cdot \left(\sum_i \alpha_i - 1 \right).$$

Compare (3.2.9) and (3.2.10) to get

$$\chi(V_{\hat{\alpha}^i}) = -\frac{\alpha_i (n-i)^6}{n^6 \left(\sum_i \alpha_i - 1 \right)} \chi(V_\alpha).$$

We now conclude by showing that the numbers $\chi(V_\alpha)/n^5$ satisfy the same recursion (3.2.4) fulfilled by the $c(\alpha)$'s. If $\alpha=(n^1)$, we know by Remark 3.2.3 that

$$\frac{1}{n^5}\chi(V_\alpha)=n.$$

For $\alpha\neq(n^1)$, we can use the above computations to find (the sums run over all indices i for which $\alpha_i\neq 0$):

$$\begin{aligned}
-\sum_i \frac{1}{(n-i)^5}\chi(V_{\hat{\alpha}^i}) &= \sum_i \frac{1}{(n-i)^5}\frac{\alpha_i(n-i)^6}{n^6\left(\sum_i \alpha_i-1\right)}\chi(V_\alpha)\\
&=\frac{1}{n^5}\frac{\sum_i \alpha_i(n-i)}{n\left(\sum_i \alpha_i-1\right)}\chi(V_\alpha)\\
&=\frac{1}{n^5}\frac{n\sum_i \alpha_i-\sum_i i\alpha_i}{n\sum_i \alpha_i-n}\chi(V_\alpha)\\
&=\frac{1}{n^5}\chi(V_\alpha).
\end{aligned}$$

Lemma 3.2.2 is proved. As noted in Section 3.2.1, this completes the proof of Theorem 3.1.1.

Remark 3.2.4. For an abelian variety X of arbitrary dimension g, Shen [72] observes that from an equality of formal power series in q,

$$\sum_{n\geq 0} P_{g-1}(n)q^n=\exp\left(\sum_{n\geq 1} s_n q^n\right),$$

defining the sequence $\{s_n\}_{n\geq 1}$, one obtains by application of the operator $q\frac{\mathrm{d}}{\mathrm{d}q}$ the identity

$$nP_{g-1}(n)=\sum_{k=1}^{n} k s_k P_{g-1}(n-k).$$

Starting with this equality, our proofs of Lemmas 3.2.1 and 3.2.2, with $\chi(V_\alpha)/n^5$ replaced by $\chi(V_\alpha)/n^{2g-1}$, go through without change, and we recover the identity (3.1.1). ◇

4 CURVE COUNTING VIA QUOT SCHEMES

This chapter is essentially the content of the paper [68].

4.1 Introduction

One of the conjectures in [50] stated that 0-dimensional Donaldson–Thomas (DT, for short) invariants of a smooth projective Calabi–Yau threefold equal the signed Euler characteristic of the moduli space. Now, the more general formula

$$\tilde{\chi}(\mathrm{Hilb}^n Y) = (-1)^n \chi(\mathrm{Hilb}^n Y) \tag{4.1.1}$$

is known to hold for *any* smooth threefold Y, proper or not [9, Thm. 4.11]. Here $\tilde{\chi} = \chi(-, \nu)$ is the Euler characteristic weighted by the Behrend function [5]. The 0-dimensional MNOP conjecture is also solved with cobordism techniques in [48, 47].

4.1.1 Main result

We propose a statement analogous to (4.1.1), again with no Calabi–Yau or properness assumption on the threefold Y, but where a curve is present. More precisely, we focus on the space of 1-dimensional subschemes $Z \subset Y$ whose fundamental class is the cycle of a fixed Cohen–Macaulay curve $C \subset Y$. A natural scheme structure on this space seems to be provided by the Quot scheme

$$Q_C^n = \mathrm{Quot}_n(\mathscr{I}_C)$$

of 0-dimensional length n quotients of $\mathscr{I}_C$, the ideal sheaf of C. By identifying a surjection $\mathscr{I}_C \twoheadrightarrow F$ with its kernel $\mathscr{I}_Z$, we see that Q_C^n parametrizes curves $Z \subset Y$ differing from C by a finite subscheme of length n. Our main result, proved in Section 4.4, is the following weighted Euler characteristic computation.

THEOREM. *Let Y be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. Then*

$$\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n). \tag{4.1.2}$$

The proof uses stratification techniques as in [9] and [6].

4.1.2 Applications

Let Y be a smooth projective threefold. Let $I_m(Y,\beta)$ be the Hilbert scheme of curves $Z \subset Y$ in class $\beta \in H_2(Y,\mathbb{Z})$, with $\chi(\mathscr{O}_Z) = m$. Given a Cohen–Macaulay curve $C \subset Y$ of arithmetic genus g, embedded in class β, we show there is a closed immersion $\iota : Q_C^n \to I_{1-g+n}(Y,\beta)$. We define

$$I_n(Y,C) \subset I_{1-g+n}(Y,\beta) = I \tag{4.1.3}$$

to be its scheme-theoretic image. When Y is Calabi–Yau, we define the *contribution* of C to the full (degree β) DT invariant of I to be the weighted Euler characteristic

$$\mathsf{DT}_{n,C} = \chi(I_n(Y,C),\nu_I). \tag{4.1.4}$$

A first consequence of (4.1.2) is the identity

$$\mathsf{DT}_{n,C} = (-1)^n\chi(I_n(Y,C))$$

when C is a smooth *rigid* curve in Y, because in this case (4.1.3) is both open and closed.

Local DT/PT correspondence

Let $P_m(Y,\beta)$ be the moduli space of stable pairs introduced by Pandharipande and Thomas [62]. For a Calabi–Yau threefold Y and a homology class $\beta \in H_2(Y,\mathbb{Z})$, the generating functions encoding the DT and PT invariants of Y satisfy the "wall-crossing type" formula

$$\mathsf{DT}_\beta(Y,q) = M(-q)^{\chi(Y)}\cdot \mathsf{PT}_\beta(Y,q).$$

Here and throughout, $M(q)$ denotes the MacMahon function, the generating series of plane partitions, that is,

$$M(q) = \sum_\pi q^{|\pi|} = \prod_{k\geq 1}(1-q^k)^{-k}.$$

The DT/PT correspondence stated above was first conjectured in [62] and later proved in [15, 79]. In this paper we ask about a similar formula relating the *local* invariants, that is, the contributions of a single smooth curve $C \subset Y$ to the full DT and PT invariants of Y in the class $\beta = [C]$.

If $C \subset Y$ is a fixed smooth curve of genus g, we consider the closed subscheme

$$P_n(Y,C) \subset P_{1-g+n}(Y,\beta) = P$$

of stable pairs with Cohen–Macaulay support equal to C. We use (4.1.2) and the isomorphism $P_n(Y,C) \cong \operatorname{Sym}^n C$ to show the generating function identity

$$\sum_{n\geq 0}\tilde{\chi}(I_n(Y,C))q^n = M(-q)^{\chi(Y)}(1+q)^{2g-2}, \tag{4.1.5}$$

which holds without any Calabi–Yau assumption.

For Y a Calabi–Yau threefold, we consider the stable pair local contributions

$$\mathsf{PT}_{n,C} = \chi(P_n(Y,C),\nu_P)$$

like we did in (4.1.4) for ideal sheaves. We assemble all the local invariants into generating functions

$$\mathsf{DT}_C(q) = \sum_{n\geq 0} \mathsf{DT}_{n,C}\, q^n$$
$$\mathsf{PT}_C(q) = \sum_{n\geq 0} \mathsf{PT}_{n,C}\, q^n.$$

The PT side has been computed [63, Lemma 3.4] and the result is

$$\mathsf{PT}_C(q) = n_{g,C}\cdot (1+q)^{2g-2},$$

where $n_{g,C}$ is the BPS number of C. Therefore it is clear by looking at (4.1.5) that the DT/PT correspondence

$$\mathsf{DT}_C(q) = M(-q)^{\chi(Y)}\cdot \mathsf{PT}_C(q) \tag{4.1.6}$$

holds for C if and only if, for every n, one has

$$\mathsf{DT}_{n,C} = n_{g,C}\cdot \tilde{\chi}(I_n(Y,C)).$$

For instance, it holds when C is rigid. In the last section, we discuss the plausibility to conjecture the identity (4.1.6) to hold for all smooth curves.

Conventions. In this paper, all schemes are defined over $\mathbb{C}$, and all threefolds are assumed to be smooth. An *ideal sheaf* is a torsion-free sheaf with rank one and trivial determinant. For a smooth projective threefold Y, we denote by $I_m(Y,\beta)$ the moduli space of ideal sheaves with Chern character $(1,0,-\beta,-m+\beta\cdot c_1(Y)/2)$. It is naturally isomorphic to the Hilbert scheme parametrizing closed subschemes $Z\subset Y$ of codimension at least 2, with homology class β and $\chi(\mathscr{O}_Z)=m$. A *Cohen–Macaulay* curve is a scheme of *pure* dimension one without embedded points. The *Calabi–Yau* condition for us is simply the existence of a trivialization $\omega_Y\cong\mathscr{O}_Y$. We use the word *rigid* as a shorthand for the more correct *infinitesimally rigid*: for a smooth embedded curve $C\subset Y$, this means $H^0(C,N_{C/Y})=0$, where $N_{C/Y}$ is the normal bundle. Finally, we refer to [5] for the main properties of the Behrend function and of the weighted Euler characteristic

4.2 The local model

The global geometry of a fixed smooth curve in a threefold $C\subset Y$ will be analysed through the local model

$$\mathbb{A}^1\subset\mathbb{A}^3$$

of a line in affine space. We get started by introducing the moduli space of ideal sheaves for this local model.

Let X be the resolved conifold, that is, the total space of the rank two bundle

$$\mathscr{O}_{\mathbb{P}^1}(-1,-1) \to \mathbb{P}^1.$$

It is a quasi-projective Calabi–Yau threefold. We let $C_0 \subset X$ be the zero section, and $\mathbb{A}^3 \subset X$ a *fixed* chart of the bundle.

Definition 4.2.1. For any integer $n \geq 0$, we define

$$M_n \subset I_{n+1}(X, [C_0])$$

to be the open subscheme parametrizing ideal sheaves $\mathscr{I}_Z \subset \mathscr{O}_X$ such that no associated point of Z is contained in $X \setminus \mathbb{A}^3$. ♢

Since C_0 is rigid, we can interpret M_n as the moduli space of "curves" in $\mathbb{A}^3$, consisting of a *fixed* affine line $L = C_0 \cap \mathbb{A}^3$ together with n roaming points.

The scheme M_n seems to be the perfect local playground for studying the enumerative geometry of a fixed curve (with n points) in a threefold. Exactly like studying $\operatorname{Hilb}^n \mathbb{A}^3$ was essential [9] to unveil the Donaldson–Thomas theory of $\operatorname{Hilb}^n Y$, where Y is any Calabi–Yau threefold, the space M_n will help us to figure out the DT contribution of a fixed smooth rigid curve in a Calabi–Yau threefold (and, conjecturally, all smooth curves). Forgetting about the Calabi–Yau assumption, we will find out that understanding the local picture in $\mathbb{A}^3$ gives information about *arbitrary* threefolds, in perfect analogy with the results of [9].

In the rest of this section, we show that M_n is isomorphic to the Quot scheme of the ideal sheaf of a line, and we compute its DT invariant via equivariant localization.

Let L denote the line $C_0 \cap \mathbb{A}^3$. Note that if $Z \subset X$ corresponds to a point of M_n, by definition its embedded points can only be supported on L. Similarly, isolated points are confined to the chart $\mathbb{A}^3 \subset X$.

Proposition 4.2.2. *There is an isomorphism of schemes* $M_n \cong \operatorname{Quot}_n(\mathscr{I}_L)$.

Proof. Let T be a scheme and let $\iota : \mathbb{A}^3 \times T \to X \times T$ be the natural open immersion. If $\mathscr{O}_{X\times T} \twoheadrightarrow \mathscr{O}_{\mathcal{Z}}$ represents a T-valued point of M_n, we can consider the sheaf $\mathscr{F} = \mathscr{I}_{C_0 \times T} / \mathscr{I}_{\mathcal{Z}}$, which by definition of M_n is supported on a subscheme of $\mathbb{A}^3 \times T$ which is finite of relative length n over T. Restricting the short exact sequence

$$0 \to \mathscr{F} \to \mathscr{O}_{\mathcal{Z}} \to \mathscr{O}_{C_0 \times T} \to 0$$

to $\mathbb{A}^3 \times T$ gives a short exact sequence

$$0 \to \iota^*\mathscr{F} \to \iota^*\mathscr{O}_{\mathcal{Z}} \to \mathscr{O}_{L\times T} \to 0$$

with T-flat kernel, so we get a T-valued point $\mathscr{I}_{L\times T} \twoheadrightarrow \iota^*\mathscr{F}$ of $\operatorname{Quot}_n(\mathscr{I}_L)$, since as we already noticed $\iota^*\mathscr{F}$ has the same support as $\mathscr{F}$.

Conversely, a T-flat quotient $\mathscr{F}$ of the ideal sheaf $\mathscr{I}_{L\times T}$ determines a flat family of subschemes

$$\mathcal{Z} \subset \mathbb{A}^3 \times T \to T,$$

where $L \times T \subset \mathcal{Z}$. Taking closures inside $X \times T$, we get closed immersions

$$C_0 \times T \subset \overline{\mathcal{Z}} \subset X \times T.$$

The support of $\mathscr{F}$ is proper over T, and since $\mathbb{A}^3$ and X are separated, we see that the inclusion maps of $\operatorname{Supp}\mathscr{F}$ in $\mathbb{A}^3 \times T$ and $X \times T$ are proper. This says that the pushforward $\iota_*\mathscr{F}$ is a coherent sheaf on $X \times T$. It agrees with the relative ideal of the immersion $C_0 \times T \subset \overline{\mathcal{Z}}$, and is supported exactly where $\mathscr{F}$ is. Finally, the short exact sequence

$$0 \to \iota_*\mathscr{F} \to \mathscr{O}_{\overline{\mathcal{Z}}} \to \mathscr{O}_{C_0\times T} \to 0$$

says $\mathscr{O}_{\overline{\mathcal{Z}}}$ is T-flat (being an extension of T-flat sheaves), therefore we get a T-valued point of M_n. The two constructions are inverse to each other, whence the claim. □

Keeping the above result in mind, we will sometimes silently identify M_n with $\operatorname{Quot}_n(\mathscr{I}_L)$, and we will switch from subschemes (or ideal sheaves) to quotient sheaves with no further mention.

Remark 4.2.3. The resolved conifold X plays little role here. In fact, the above proof shows the following. If there is an immersion $\mathbb{A}^3 \to Y$ into some Calabi–Yau threefold Y, such that the closure of a line $L \subset \mathbb{A}^3$ becomes a rigid rational curve $C \subset Y$, then the Hilbert scheme $I_{n+1}(Y,[C])$ contains an open subscheme isomorphic to $\operatorname{Quot}_n(\mathscr{I}_L)$. ♢

4.2.1 The DT invariant

The open subscheme $M_n \subset I_{n+1}(X,[C_0])$ inherits, by restriction, a torus-equivariant symmetric obstruction theory, and therefore an equivariant virtual fundamental class

$$[M_n]^{\mathrm{vir}} \in A_0^{\mathbf{T}}(M_n) \otimes \mathbb{Q}(s_1,s_2,s_3).$$

The torus $\mathbf{T} \subset (\mathbb{C}^\times)^3$ we are referring to is the two-dimensional torus fixing the Calabi–Yau form on X, and acting on X by rescaling coordinates. We refer the reader to [6, Section 2.3] for more details on this action and for an accurate description of the fixed locus

$$I_m(X,d[C_0])^{\mathbf{T}} \subset I_m(X,d[C_0])$$

for every $d > 0$. An ideal sheaf $\mathscr{I}_Z \in M_n$ is $\mathbf{T}$-fixed if it becomes a monomial ideal when restricted to the chosen chart $\mathbb{A}^3 \subset X$. The fixed locus $M_n^{\mathbf{T}} \subset M_n$ is isolated and reduced, by [50, Lemma 6 and 8]. In the language of the topological vertex, a $\mathbf{T}$-fixed ideal can be described as a way of stacking n boxes in the corner of the one-legged configuration $(\emptyset,\emptyset,\square)$. We give an example in Figure 1.

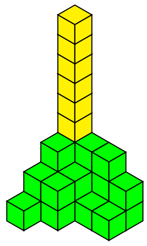


Figure 1: A $\mathbf{T}$-fixed ideal in M_n. The "z-axis" has to be figured as infinitely long, corresponding to the line $L = C_0 \cap \mathbb{A}^3$.

The parity of the tangent space dimension at $\mathbf{T}$-fixed points of $I_m(X, d[C_0])$ was computed in [6, Prop. 2.7]. The result is $(-1)^{m-d}$ by an application of [50, Thm. 2]. In our case $m = n+1$ and $d = 1$ so we get the sign $(-1)^n$ for $I_{n+1}(X,[C_0])$. Since M_n is open in this Hilbert scheme, the parity does not change and we deduce that

$$(-1)^{\dim T_{M_n}|_{\mathscr{I}}} = (-1)^n$$

for all fixed points $\mathscr{I} \in M_n^{\mathbf{T}}$. After the Calabi–Yau specialization $s_1 + s_2 + s_3 = 0$ of the equivariant parameters, and by the symmetry of the obstruction theory, the virtual localization formula [30] reads

$$[M_n]^{\mathrm{vir}} = (-1)^n [M_n^{\mathbf{T}}] \in A_0(M_n), \tag{4.2.1}$$

where, as mentioned above, the sign

$$(-1)^n = \frac{e^{\mathbf{T}}(\mathrm{Ext}^2(\mathscr{I},\mathscr{I}))}{e^{\mathbf{T}}(\mathrm{Ext}^1(\mathscr{I},\mathscr{I}))} \in \mathbb{Q}(s_1, s_2, s_3)$$

comes from [50, Thm. 2].

We define the Donaldson–Thomas invariant of M_n by equivariant localization through formula (4.2.1). Hence we can compute it as

$$\mathsf{DT}(M_n) = (-1)^n \chi(M_n),$$

where the Euler characteristic $\chi(M_n)$ counts the number of fixed points.

It is easy to see (see for instance the proof of [6, Lemma 2.9]) that

$$\sum_{n\geq 0} \chi(M_n) q^n = \frac{M(q)}{1-q} \tag{4.2.2}$$

where $M(q) = \prod_{m\geq 1}(1-q^m)^{-m}$ is the MacMahon function, the generating series of plane partitions. In particular, the DT partition function for the moduli spaces M_n takes the form

$$\sum_{n\geq 0} \mathsf{DT}(M_n) q^{n+1} = q\frac{M(-q)}{1+q} = q(1-2q+5q^2-11q^3+\cdots).$$

In the sum, we have switched indices by one to follow the general convention of weighting the variable q by the holomorphic Euler characteristic.

4.3 Curves and Quot schemes

4.3.1 Main characters

Let C be a Cohen–Macaulay curve embedded in a quasi-projective variety Y and let $\mathscr{I}_C \subset \mathscr{O}_Y$ denote its ideal sheaf. For an integer $n \geq 0$, let $Q = \mathrm{Quot}_n(\mathscr{I}_C)$ be the Quot scheme parametrizing 0-dimensional quotients of $\mathscr{I}_C$, of length n. See [57] for a proof of the representability of the Quot functor in the quasi-projective case. By looking at the full exact sequence

$$0 \to \mathscr{I}_Z \to \mathscr{I}_C \to F \to 0$$

for a given point $[\mathscr{I}_C \twoheadrightarrow F]$ of Q, we think of the Quot scheme as parametrizing curves $Z \subset Y$ obtained from C, roughly speaking, by adding a finite subscheme of length n.

Definition 4.3.1. We denote by $W_C^n \subset Q$ the closed subset parametrizing quotients $\mathscr{I}_C \twoheadrightarrow F$ such that $\mathrm{Supp}\, F \subset C$, where $\mathrm{Supp}\, F$ denotes the *set-theoretic* support of the sheaf F. We endow W_C^n with the reduced scheme structure. $\diamondsuit$

Given a point $[F] \in W_C^n$, the support of F has the structure of a closed subscheme of Y but not of C in general; however, $\mathrm{Supp}\, F$ defines naturally an effective zero-cycle on C. Sending $[F]$ to this cycle is a morphism, as we now show.

LEMMA 4.3.2. *There is a natural morphism $u : W_C^n \to \mathrm{Sym}^n C$ sending a quotient to the corresponding zero-cycle.*

PROOF. Let T be a reduced scheme, which we take as the base of a valued point $\mathscr{I}_{C\times T} \twoheadrightarrow \mathscr{F}$ of W_C^n. Let $\pi : Y \times T \to T$ be the projection. Working locally on Y and T we see that by Nakayama's lemma, $\mathrm{Supp}\, \mathscr{F} \cap \pi^{-1}(t) = \mathrm{Supp}\, \mathscr{F}_t$ for every closed point $t \in T$. Then the closed subscheme $\mathrm{Supp}\, \mathscr{F} \subset Y \times T$ is flat over T (because the Hilbert polynomial of the fibres $\mathrm{Supp}\, \mathscr{F}_t$ is the constant n and T is reduced), and hence defines a valued point $T \to \mathrm{Hilb}^n Y$. Composing with the Hilbert-Chow map $\mathrm{Hilb}^n Y \to \mathrm{Sym}^n Y$ we get a morphism $T \to \mathrm{Sym}^n Y$ which factors through $\mathrm{Sym}^n C$, by definition of W_C^n. $\square$

For every partition $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots r^{\alpha_r})$ of $n = \sum_i i\alpha_i$ there is a locally closed subscheme

$$\mathrm{Sym}^n_\alpha C \subset \mathrm{Sym}^n C$$

parametrizing zero-cycles whose support consists of α_i points of multiplicity i, for each $i = 1, \ldots, r$. So the number of *distinct* points in the support is $\|\alpha\| = \sum_i \alpha_i$. The above subschemes form a locally closed stratification of $\mathrm{Sym}^n C$, which we can use together with the morphism u to stratify W_C^n by locally closed subschemes

$$W_C^\alpha = u^{-1}(\mathrm{Sym}^n_\alpha C) \subset W_C^n. \tag{4.3.1}$$

In particular, since $\mathrm{Sym}^n_{(n)} C \cong C$, there is a natural morphism

$$\pi_C : W_C^{(n)} \to C \tag{4.3.2}$$

corresponding to the deepest stratum.

The main result of this section asserts that, when C is a smooth curve and Y is a smooth threefold, the map (4.3.2) is a Zariski locally trivial fibration. The proof is based on the Quot scheme adaptation of the results proven by Behrend and Fantechi for $\mathrm{Hilb}^n Y$ [9, Section 4].

Let us now introduce what will turn out to be the typical fibre of π_C. Recall that X denotes the resolved conifold and $C_0 \subset X$ is the zero section.

Definition 4.3.3. We denote by $F_n \subset M_n$ the closed subset parametrizing subschemes $Z \subset X$ such that the relative ideal $\mathscr{I}_{C_0}/\mathscr{I}_Z$ is entirely supported at the origin $0 \in L = C_0 \cap \mathbb{A}^3$. We use the shorthand

$$\nu_n = \nu_{M_n}\big|_{F_n}$$

for the restriction of the Behrend function on M_n to F_n. ♢

We can think of F_n and all strata $W_C^\alpha \subset W_C^n$ as endowed with the reduced scheme structure.

Remark 4.3.4. The morphism $u : W_C^n \to \mathrm{Sym}^n C$ plays the role of the Hilbert-Chow map $\mathrm{Hilb}^n Y \to \mathrm{Sym}^n Y$ in the 0-dimensional setting, and the subscheme $F_n \subset M_n$ is the analogue of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^3)_0 \subset \mathrm{Hilb}^n \mathbb{A}^3$ parametrizing finite subschemes supported at the origin. ♢

PROPOSITION 4.3.5. *There is a natural isomorphism $W_L^{(n)} = L \times F_n$. Moreover, if $p : W_L^{(n)} \to F_n$ is the projection, we have the relation*

$$\nu_{M_n}\big|_{W_L^{(n)}} = p^* \nu_n. \tag{4.3.3}$$

PROOF. We view L as the additive group $\mathbb{G}_a$ and we let it act on itself by translation. This induces an action of L on M_n. Restricting this action to F_n gives a map

$$L \times F_n \to W_L^{(n)}.$$

This is an isomorphism, whose inverse is the morphism $\pi_L \times \rho : W_L^{(n)} \to L \times F_n$, where

$$\rho : W_L^{(n)} \to F_n$$

takes a subscheme $[Z] \in W_L^{(n)}$ to its translation by $-x \in \mathbb{G}_a$, where $x \in L = \mathbb{G}_a$ is the unique embedded point on Z. The identity (4.3.3) follows because the Behrend function is constant on orbits and for each $P \in F_n$ the slice $L \times \{P\}$ is isomorphic to an orbit. □

4.3.2 Comparing Quot schemes

Let $\varphi : Y \to Y'$ be a morphism of varieties, where Y is quasi-projective and Y' is complete. Let $C' \subset Y'$ be a Cohen–Macaulay curve and let $C = \varphi^{-1}(C') \subset Y$ denote its preimage. We assume C is a Cohen–Macaulay curve and C' is its scheme-theoretic image. In Lemma 4.3.6 we give sufficient conditions for this to hold.

Given an integer $n \geq 0$, we let $Q = \mathrm{Quot}_n(\mathscr{I}_C)$ and $Q' = \mathrm{Quot}_n(\mathscr{I}_{C'})$.

We will show how to associate to these data a rational map

$$\Phi : Q \dashrightarrow Q'.$$

The rough idea is that we would like to "push down" the n points in the support of a sheaf $[F] \in Q$ and still get n points, which would ideally form the support of the image sheaf $\varphi_* F$. This only works, as one might expect, over the open subscheme $V \subset Q$ parametrizing sheaves F such that $\varphi|_{\mathrm{Supp}\, F}$ is injective. Moreover, the resulting map $\Phi : V \to Q'$ turns out to be étale whenever φ is. After extending this result to quasi-projective Y', we will be able to compare $\mathrm{Quot}_n(\mathscr{I}_C)$ with the local picture of $M_n = \mathrm{Quot}_n(\mathscr{I}_L)$, and pull back (étale-locally) the known results about π_L (Proposition 4.3.5) to deduce that the maps π_C defined in (4.3.2) are Zariski locally trivial, at least when C and Y are smooth.

Lemma 4.3.6. *Let $\varphi : Y \to Y'$ be an étale morphism of varieties with image U. If $C' \subset Y'$ is a Cohen–Macaulay curve and $U \cap C'$ is dense in C', then $C = \varphi^{-1}(C')$ is Cohen–Macaulay and C' is its scheme-theoretic image.*

Before proving the lemma, recall that a closed subscheme C' of a scheme Y' is said to have an embedded component if there is a dense open subset $U \subset Y'$ such that $U \cap C'$ is dense in C' but its scheme-theoretic closure does not equal C' scheme-theoretically. Recall that a curve is Cohen–Macaulay if it has no embedded points.

Proof. Since the restriction $C \to C'$ is étale and C' is Cohen–Macaulay, C is also Cohen–Macaulay. Moreover, U is open (because φ is étale) and dense (because Y' is irreducible), and since $U \cap C' \subset C'$ is dense, the scheme-theoretic closure of $U \cap C'$ agrees with C' topologically. But since C' has no embedded points, they in fact agree as schemes. On the other hand, the open subset $U \cap C' \subset C'$ is the set-theoretic image of the étale map $C \to C'$. Therefore its scheme-theoretic closure is the scheme-theoretic image of $C \to C'$. So C' is the scheme-theoretic image of C. □

Notation. For a scheme S, we will denote $\varphi_S = \varphi \times \mathrm{id}_S : Y \times S \to Y' \times S$. The case $S = Q$ being quite special, we will let $\tilde{\varphi}$ denote $\varphi_Q = \varphi \times \mathrm{id}_Q$.

By our assumptions, $C' \times S$ is the scheme-theoretic image of $C \times S \subset Y \times S$ under φ_S, for any scheme S. Indeed, φ is quasi-compact so the scheme-theoretic image commutes with flat base change.

Remark 4.3.7. Let $\mathscr{E}$ be the universal sheaf on Q, with scheme-theoretic support $\Sigma \subset Y \times Q$. Since $\Sigma \to Q$ is proper (by the very definition of the Quot functor), and it factors through the (separated) projection $\pi : Y' \times Q \to Q$, necessarily the map $\Sigma \to Y' \times Q$ must be proper. Since $\tilde{\varphi}_*\mathscr{E}$ is obtained as a pushforward from Σ, it is coherent. Therefore, pushing forward coherent sheaves supported on Σ will still give us coherent sheaves, even if φ is not proper. $\diamondsuit$

Let $[F] \in Q$ be any point, and let $\mathscr{I}_Z \subset \mathscr{I}_C$ be the kernel of the surjection. Then we have closed immersions $C \subset Z \subset Y$ and $C' \subset Z' \subset Y'$, where Z' denotes the scheme-theoretic image of Z. Using that $R^1\varphi_*F = 0$, we find a commutative diagram of coherent $\mathscr{O}_{Y'}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}_{C'}/\mathscr{I}_{Z'} \arrow[r] \arrow[d, hook] & \mathscr{O}_{Z'} \arrow[r] \arrow[d, hook] & \mathscr{O}_{C'} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \varphi_*F \arrow[r] & \varphi_*\mathscr{O}_Z \arrow[r] & \varphi_*\mathscr{O}_C \arrow[r] & 0 \end{tikzcd}$$

having exact rows. The middle and right vertical arrows are monomorphisms by definition of scheme-theoretic image. For instance,

$$\mathscr{I}_{C'} = \ker\big(\mathscr{O}_{Y'} \twoheadrightarrow \mathscr{O}_{C'}\big) = \ker\big(\mathscr{O}_{Y'} \twoheadrightarrow \mathscr{O}_{C'} \to \varphi_*\mathscr{O}_C\big)$$

implies that $\mathscr{O}_{C'} \to \varphi_*\mathscr{O}_C$ is injective.

In fact, this observation can be made universal. Let $\mathscr{I}_{C\times Q} \twoheadrightarrow \mathscr{E}$ be the universal quotient, living over $Y \times Q$. Looking at its kernel $\mathscr{I}_{\mathcal{Z}}$, we get a commutative diagram

$$\begin{tikzcd} C \times Q \arrow[r, hook] \arrow[d] & \mathcal{Z} \arrow[r, hook] \arrow[d] & Y \times Q \arrow[d, "\tilde{\varphi}"] \\ C' \times Q \arrow[r, hook] & \mathcal{Z}' \arrow[r, hook] & Y' \times Q \end{tikzcd}$$

where the horizontal arrows are closed immersions, $\tilde{\varphi} = \varphi \times \mathrm{id}_Q$ and $\mathcal{Z}'$ denotes the scheme-theoretic image of $\mathcal{Z}$. We also get a commutative diagram of coherent $\mathscr{O}_{Y'\times Q}$-modules

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}_{C'\times Q}/\mathscr{I}_{\mathcal{Z}'} \arrow[r] \arrow[d, hook] & \mathscr{O}_{\mathcal{Z}'} \arrow[r] \arrow[d, hook] & \mathscr{O}_{C'\times Q} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \tilde{\varphi}_*\mathscr{E} \arrow[r] & \tilde{\varphi}_*\mathscr{O}_{\mathcal{Z}} \arrow[r] & \tilde{\varphi}_*\mathscr{O}_{C\times Q} \arrow[r] & 0 \end{tikzcd}$$

having exact rows.

Let us consider the composition

$$\alpha : \mathscr{I}_{C'\times Q} \twoheadrightarrow \mathscr{I}_{C'\times Q}/\mathscr{I}_{\mathcal{Z}'} \hookrightarrow \tilde{\varphi}_*\mathscr{E} \tag{4.3.4}$$

and let us write $\mathscr{K}$ for its cokernel. By Remark 4.3.7, $\tilde{\varphi}_*\mathscr{E}$ is coherent, hence $\mathscr{K} = \operatorname{coker}\alpha$ is coherent, too. Thus $\operatorname{Supp}\mathscr{K}$ is closed in $Y' \times Q$. Since Y' is complete, the projection $\pi : Y' \times Q \to Q$ is closed. Therefore the complement

$$Q \setminus \pi(\operatorname{Supp}\mathscr{K}) \subset Q \tag{4.3.5}$$

is an open subset of Q.

PROPOSITION 4.3.8. *Let* $[F] \in Q$ *be a point such that* φ *is étale in a neighborhood of* $\operatorname{Supp} F$ *and* $\varphi(x) \neq \varphi(y)$ *for all distinct points* $x, y \in \operatorname{Supp} F$*. Then there is an open neighborhood* $U \subset Q$ *of* $[F]$ *admitting an étale map* $\Phi : U \to Q'$*.*

PROOF. We first observe that we may reduce to prove the result after restricting Y to *any* open neighborhood of $\operatorname{Supp} F$ inside Y. Indeed, if V is any such neighborhood, $\operatorname{Quot}_n(\mathscr{I}_C|_V)$ is an open subscheme of Q that still contains $[F]$ as a point. We will take advantage of this freedom by choosing a suitable V. We divide the proof in two steps.

Step 1: *Existence of the map.* Let $Z \subset Y$ be the closed subscheme determined by the kernel of $\mathscr{I}_C \twoheadrightarrow F$. Let $Z' \subset Y'$ be its scheme-theoretic image. Since $\varphi|_{\operatorname{Supp} F}$ is injective and φ is étale around $\operatorname{Supp} F$, the natural monomorphism $\mathscr{I}_{C'}/\mathscr{I}_{Z'} \to \varphi_* F$ is an isomorphism and $\varphi_* F$ is a sheaf of length n, so that we get a well-defined point

$$[\varphi_* F] \in Q'. \tag{4.3.6}$$

Now let $B \subset Y$ denote the support of F and let V be an open neighborhood of B such that φ is étale when restricted to V. We may assume V is affine, and in fact we may also assume $Y = V$, by our initial remark.

In this situation, we have the cartesian square

$$\begin{tikzcd} Y \times [F] \arrow[r, hook, "i"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & Y \times Q \arrow[d, "\tilde{\varphi}"] \\ Y' \times [F] \arrow[r, hook, "j"] & Y' \times Q \end{tikzcd}$$

where the map $\tilde{\varphi}$ is affine (as now Y is affine). Therefore, working affine-locally on $Y' \times Q$, we see that the natural base change map $j^* \tilde{\varphi}_* \mathscr{E} \xrightarrow{\sim} \varphi_* F$ is an isomorphism. This proves that the surjection $\mathscr{I}_{C'} \twoheadrightarrow \varphi_* F$ defining the point (4.3.6) is obtained precisely restricting $\alpha : \mathscr{I}_{C' \times Q} \to \tilde{\varphi}_* \mathscr{E}$, defined in (4.3.4), to the slice

$$j : Y' \times [F] \subset Y' \times Q.$$

Letting $U \subset Q$ denote the open subset defined in (4.3.5), we see that α restricts to a surjection

$$\alpha|_{Y' \times U} : \mathscr{I}_{C' \times U} \twoheadrightarrow \varphi_{U*} \mathscr{E}_U,$$

where $\mathscr{E}_U = \mathscr{E}|_{Y \times U}$. The target is a coherent sheaf, and it is flat over U. Indeed, $\mathscr{E}$ is flat over Q, thus $\tilde{\varphi}_* \mathscr{E}$ is also flat over Q. But $\varphi_{U*} \mathscr{E}_U$ is naturally isomorphic to the pullback of $\tilde{\varphi}_* \mathscr{E}$ along the open immersion $Y' \times U \subset Y' \times Q$, therefore it is flat over U. Finally, the map $\alpha|_{Y' \times U}$ restricts to length n quotients

$$\mathscr{I}_{C'} \twoheadrightarrow \varphi_* E,$$

for any closed point $[E] \in U$. Therefore we have just constructed a morphism

$$\Phi : U \to Q', \qquad [E] \mapsto [\varphi_* E].$$

Step 2: *Proving it is étale.* We may shrink Y further and replace it by any affine open neighborhood of $B = \operatorname{Supp} F$ contained in $Y \setminus A$, where A is the closed subset

$$A = \coprod_{b \in B} \varphi^{-1}\varphi(b) \setminus \{b\} \subset Y.$$

After this choice, the preimage $Y_{\varphi(b)}$ is the single point $\{b\}$, for every $b \in B$. This condition implies that the natural morphism

$$\varphi^*\varphi_*F \xrightarrow{\sim} F \tag{4.3.7}$$

is an isomorphism. Although this condition is not preserved in any open neighborhood of $[F]$, it is preserved infinitesimally, which is exactly what we need to establish étaleness.

We now use the infinitesimal criterion to show Φ is étale at the point $[F]$. Let $\iota : T \to \overline{T}$ be a small extension of fat points. Assume we have a commutative square

$$\begin{tikzcd} T \arrow[r, hook, "\iota"] \arrow[d, "g"'] & \overline{T} \arrow[d, "h"] \arrow[dl, dotted, "v"] \\ U \arrow[r, "\Phi"] & Q' \end{tikzcd}$$

where g sends the closed point $0 \in T$ to $[F]$. Then we want to find a *unique* arrow v making the two induced triangles commutative. Rephrasing this in terms of families of sheaves, let $\mathscr{I}_{C \times T} \twoheadrightarrow \mathscr{G}$ and $\mathscr{I}_{C' \times \overline{T}} \twoheadrightarrow \mathscr{H}$ be the families corresponding to g and h, living over $Y \times T$ and $Y' \times \overline{T}$ respectively. We are after a unique U-valued family $\mathscr{I}_{C \times \overline{T}} \twoheadrightarrow \mathscr{V}$ over $Y \times \overline{T}$ with the following properties.

($\star$) The condition $\Phi \circ v = h$ means we can find a commutative diagram

$$\begin{tikzcd} \mathscr{I}_{C' \times \overline{T}} \arrow[r, two heads] \arrow[d, equal] & \varphi_{\overline{T}*}\mathscr{V} \arrow[d, "\wr"] \\ \mathscr{I}_{C' \times \overline{T}} \arrow[r, two heads] & \mathscr{H} \end{tikzcd} \qquad \text{of sheaves on } Y' \times \overline{T}.$$

Let us explain the condition in detail. We use, in the following, the notation $\tilde{p} = 1_Y \times p$ and $\overline{p} = 1_{Y'} \times p$, for a given map p. Looking at the diagram

$$\begin{tikzcd} Y \times \overline{T} \arrow[r, "\varphi_{\overline{T}}"] \arrow[d, "\tilde{v}"] & Y' \times \overline{T} \arrow[d, "\overline{v}"] \\ Y \times U \arrow[r, "\varphi_U"] \arrow[d, hook] & Y' \times U \arrow[d, "\overline{\Phi}"] \\ Y \times Q & Y' \times Q' \end{tikzcd}$$

we should require

$$\mathscr{H} \cong \overline{v}^*\overline{\Phi}^*\mathscr{E}',$$

where $\mathscr{E}'$ is the universal quotient sheaf on $Y' \times Q'$. However,

$$\overline{v}^*\overline{\Phi}^*\mathscr{E}' \cong \overline{v}^*\varphi_{U*}\mathscr{E}_U \cong \varphi_{\overline{T}*}\mathscr{V},$$

where we have used "affine base change" again.

$(\star\star)$ Looking at

$$\begin{tikzcd} Y \times T \arrow[r, "\varphi_T"] \arrow[d, hook, "\tilde{\iota}"'] \arrow[dr, phantom, "\square"] & Y' \times T \arrow[d, hook, "\bar{\iota}"] \\ Y \times \overline{T} \arrow[r, "\varphi_{\overline{T}}"] & Y' \times \overline{T}, \end{tikzcd}$$

the condition $v \circ \iota = g$ means we can find a commutative diagram

$$\begin{tikzcd} \tilde{\iota}^*\mathscr{I}_{C\times\overline{T}} \arrow[r, two heads] \arrow[d, equal] & \tilde{\iota}^*\mathscr{V} \arrow[d, "\wr"] \\ \mathscr{I}_{C\times T} \arrow[r, two heads] & \mathscr{G} \end{tikzcd} \qquad \text{of sheaves on } Y \times T.$$

We observe that

(i) the isomorphism $\varphi_{\overline{T}*}\mathscr{V} \xrightarrow{\sim} \mathscr{H}$ defining $(\star)$, and

(ii) the isomorphism $\varphi_{\overline{T}}^*\varphi_{\overline{T}*}\mathscr{V} \xrightarrow{\sim} \mathscr{V}$, the "infinitesimal thickening" of (4.3.7),

together determine v uniquely: it is the *unique* arrow corresponding to the isomorphism class of the surjection

$$\mathscr{I}_{C\times\overline{T}} = \varphi_{\overline{T}}^*\mathscr{I}_{C'\times\overline{T}} \twoheadrightarrow \varphi_{\overline{T}}^*\mathscr{H} = \mathscr{V}.$$

To check that condition $(\star\star)$ is fulfilled by this family, we use that $\Phi \circ g = h \circ \iota$. In other words, there is a commutative diagram

$$\begin{tikzcd} \bar{\iota}^*\mathscr{I}_{C'\times\overline{T}} \arrow[r, two heads] \arrow[d, equal] & \bar{\iota}^*\mathscr{H} \arrow[d, "\wr"] \\ \mathscr{I}_{C'\times T} \arrow[r, two heads] & \varphi_{T*}\mathscr{G} \end{tikzcd} \qquad \text{of sheaves on } Y' \times T.$$

As before, we have noted that the family corresponding to $\Phi \circ g$ is

$$\overline{g}^*\varphi_{U*}\mathscr{E}_U \cong \varphi_{T*}\mathscr{G},$$

where $\overline{g}$ is the map $\mathrm{id}_{Y'} \times g : Y' \times T \to Y' \times U$. Now we can compute

$$\tilde{\iota}^*\mathscr{V} = \tilde{\iota}^*\varphi_{\overline{T}}^*\mathscr{H} \cong \varphi_T^*\bar{\iota}^*\mathscr{H} \cong \varphi_T^*\varphi_{T*}\mathscr{G} \cong \mathscr{G}.$$

This finishes the proof. $\square$

COROLLARY 4.3.9. *Let $\varphi : Y \to Y'$ be an étale map of quasi-projective varieties, $C' \subset Y'$ a Cohen–Macaulay curve with preimage C. Let $V \subset Q$ be the open subset parametrizing quotients $\mathscr{I}_C \twoheadrightarrow F$ such that $\varphi(x) \neq \varphi(y)$ for all $x \neq y \in \operatorname{Supp} F$. Then there is an étale map $\Phi : V \to Q'$.*

PROOF. To apply Proposition 4.3.8, we need the target to be complete. Therefore, after completing Y' to a proper variety $\overline{Y'}$, let us denote by $\overline{C'}$ the scheme-theoretic closure of C'. Then, Proposition 4.3.8 gives us an étale map $\Phi : V \to \overline{Q'}$, where the target is the scheme of length n quotients of $\mathscr{I}_{\overline{C'}}$. The map sends $[F] \mapsto [\iota_* \varphi_* F]$, where $\iota : Y' \to \overline{Y'}$ is the open immersion. However, the support of $\iota_* \varphi_* F$ can be identified with $\operatorname{Supp}(\varphi_* F) \subset Y'$ for all $[F]$, so that Φ actually factors through Q'. $\square$

4.3.3 Applications to threefolds

In this section we assume Y and Y' are quasi-projective threefolds. All the other assumptions and notations from the previous sections remain unchanged here.

If $\varphi : Y \to Y'$ is an étale map, we see that the induced morphism

$$\Phi : V \to Q'$$

of Corollary 4.3.9, when restricted to the closed stratum $W_C^{(n)} \subset V$, appears in a Cartesian diagram

$$\begin{tikzcd} W_C^{(n)} \arrow[r, "\pi_C"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\square"] & C \arrow[d, "\varphi"] \\ W_{C'}^{(n)} \arrow[r, "\pi_{C'}"] & C' \end{tikzcd} \tag{4.3.8}$$

where the horizontal maps were defined in (4.3.2). Let $V' \subset Q'$ be the image of the étale map $\Phi : V \to Q'$. Then the commutative diagram

$$\begin{tikzcd} W_C^{(n)} \arrow[r, hook] \arrow[d, "\Phi"'] & V \arrow[r, hook, "\text{open}"] \arrow[d, "\text{ét}"] & Q \\ W_{C'}^{(n)} \arrow[r, hook] & V' \arrow[r, hook, "\text{open}"] & Q' \end{tikzcd}$$

yields the relation

$$\nu_Q\big|_{W_C^{(n)}} = \Phi^*\big(\nu_{Q'}\big|_{W_{C'}^{(n)}}\big), \tag{4.3.9}$$

which will be useful in the next proof.

PROPOSITION 4.3.10. *Let $\varphi : Y \to \mathbb{A}^3$ be an étale map of quasi-projective threefolds, and let $L \subset \mathbb{A}^3$ be a line.*

(i) If $C = \varphi^{-1}(L) \subset Y$, we have a natural isomorphism $W_C^{(n)} = C \times F_n$.

(ii) The restricted Behrend function $\nu_Q\big|_{W_C^{(n)}}$ agrees with the pullback of ν_n under the natural projection to F_n.

PROOF. With the help of (4.3.8), we find a diagram

$$\begin{tikzcd} & W_C^{(n)} \arrow[r, "\pi_C"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\square"] & C \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\text{ét}"] \\ F_n & W_L^{(n)} \arrow[l, "p"'] \arrow[r, "\pi_L"] & L \arrow[r, hook] & \mathbb{A}^3 \end{tikzcd}$$

so that the first claim follows by the isomorphism $W_L^{(n)} = L \times F_n$ of Proposition 4.3.5. As for Behrend functions, we have, using (4.3.9) and (4.3.3),

$$\nu_Q\big|_{W_C^{(n)}} = \Phi^*\big(\nu_{M_n}\big|_{W_L^{(n)}}\big) = \Phi^*\big(p^*\nu_n\big).$$

The claim follows. $\square$

The following can be viewed as the analogue of [9, Cor. 4.9].

COROLLARY 4.3.11. *Let Y be a smooth quasi-projective threefold. If $C \subset Y$ is a smooth curve, the map*

$$\pi_C : W_C^{(n)} \to C$$

is a Zariski locally trivial fibration with fibre F_n. More precisely, there exists a Zariski open covering $C_i \subset C$ such that for all i one has an isomorphism

$$(\pi_C^{-1}(C_i), \nu_Q) \cong (C_i, 1) \times (F_n, \nu_n) \tag{4.3.10}$$

of schemes with constructible functions on them.

PROOF. Cover Y with open affine subschemes U_i such that, for each i, the closed immersion $C_i = C \cap U_i \subset U_i$ is given, when C_i is nonempty, by the vanishing of two equations. We can do this because C is a local complete intersection. Possibly after shrinking each U_i, we can find étale maps $U_i \to \mathbb{A}^3$ and (using the smoothness of C) Cartesian diagrams

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U_i \arrow[d, "\text{ét}"] \\ L \arrow[r, hook] & \mathbb{A}^3 \end{tikzcd}$$

where L is a fixed line in $\mathbb{A}^3$. Combining (4.3.8) with (both statements of) Proposition 4.3.10 yields Cartesian diagrams

$$\begin{tikzcd} C_i \times F_n \arrow[r, "\pi_{C_i}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & C_i \arrow[d, hook] \\ W_C^{(n)} \arrow[r, "\pi_C"] & C \end{tikzcd}$$

and the claimed decomposition (4.3.10). $\square$

We end this section by observing that the geometry of the Quot scheme Q_C^n is quite difficult to analyze. For instance, it contains a copy of $\mathrm{Hilb}^n(Y \setminus C)$ as an open subscheme, and Hilbert schemes of points on threefolds are far from being fully understood. For sure, if C and Y are nonsingular, the same is true for Q_C^1, for

$$Q_C^1 = \mathrm{Bl}_C\, Y.$$

However, unlike $\mathrm{Hilb}^n X$, which is smooth in all dimensions if $n \leq 3$ (when X is smooth), the Quot scheme is already singular for $n = 2$, as the following example shows.

Example 4.3.12. We consider $M_2 = \mathrm{Quot}_2(\mathscr{I}_L)$ for a line $L \subset \mathbb{A}^3$, for instance $L = V(x, y)$. We will exhibit a singular point belonging to the torus fixed locus $M_2^{\mathbf{T}}$. First of all, from the stratification

$$M_2 = \mathrm{Hilb}^2(\mathbb{A}^3 \setminus L) \amalg \big(\mathbb{A}^3 \setminus L \times \mathbb{P}^1\big) \amalg W_L^2$$

we see that $\dim M_2 = 6$. Consider the point $[Z] \in M_2$ corresponding to

$$\mathscr{I}_Z = (x^2, y^2, xy, xz, yz) \subset \mathbb{C}[x, y, z].$$

This is depicted in Figure 2 below. We can fix a $\mathbb{C}$-linear basis $\{\overline{x}, \overline{y}\}$ of the relative ideal $\mathscr{I}_L/\mathscr{I}_Z \subset \mathscr{O}_Z$. A linear map $h \in \mathrm{Hom}_{\mathbb{A}^3}(\mathscr{I}_Z, \mathscr{I}_L/\mathscr{I}_Z) = T_{[Z]}M_2$ is described in terms of this basis as

$$\begin{aligned}
h(x^2) &= a_1\overline{x} + b_2\overline{y}\\
h(y^2) &= a_2\overline{x} + b_2\overline{y}\\
h(xy) &= a_3\overline{x} + b_3\overline{y}\\
h(xz) &= a_4\overline{x} + b_4\overline{y}\\
h(yz) &= a_5\overline{x} + b_5\overline{y}
\end{aligned}$$

along with the relations

$$\begin{aligned}
&y \cdot h(x^2) = x \cdot h(xy),\ \ z \cdot h(x^2) = x \cdot h(xz),\\
&x \cdot h(y^2) = y \cdot h(xy),\ \ z \cdot h(y^2) = y \cdot h(yz),\\
&x \cdot h(yz) = y \cdot h(xz) = z \cdot h(xy).
\end{aligned}$$

But all these relations are in fact the vacuous identity $0 = 0$, so the tangent space $T_{[Z]}M_2$ is 10-dimensional, and since $10 > 6$ we have that $[Z]$ is a singular point. $\diamondsuit$

4.4 The weighted Euler characteristic of Q_C^n

The goal of this section is to prove the following result, anticipated in the Introduction.

THEOREM 4.4.1. *Let Y be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. If $Q_C^n = \mathrm{Quot}_n(\mathscr{I}_C)$, then*

$$\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n).$$

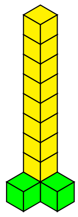


Figure 2: A singular point of the Quot scheme M_2.

4.4.1 Ingredients in the proof

We briefly discuss the main tools used in the proof of the above formula.

Stratification

We start by observing that we have a stratification

$$Q_C^n = \coprod_{\substack{0\le j\le n \\ \alpha\vdash j}} \mathrm{Hilb}^{n-j}(Y\setminus C)\times W_C^\alpha \tag{4.4.1}$$

by locally closed subschemes, "separating" the points *away from* the curve from those embedded *on* the curve. We think of a partition $\alpha\vdash j$ as a tuple of positive integers

$$\alpha_1\ge\cdots\ge\alpha_{r_\alpha}\ge 1$$

such that $\sum\alpha_i = j$. Here r_α is the number of distinct parts of α. Recall that

$$W_C^\alpha\subset Q_C^j,$$

defined for the first time in (4.3.1), parametrizes configurations of r_α distinct embedded points on C, having respective multiplicities $\alpha_1,\ldots,\alpha_{r_\alpha}$. According to (4.4.1), it is natural to expect the number

$$\tilde\chi(Q_C^n) = \chi(Q_C^n,\nu_{Q_C^n})$$

to be computed combining the following data.

First of all, "point contributions" from $\mathrm{Hilb}^{n-j}(Y\setminus C)$ are taken care of by [9, Thm. 4.11], which implies the formula

$$\tilde\chi(\mathrm{Hilb}^k(Y\setminus C)) = (-1)^k\chi(\mathrm{Hilb}^k(Y\setminus C)). \tag{4.4.2}$$

Secondly, contributions from $W_C^\alpha\subset W_C^j$ will be fully expressed (thanks to the content of the previous section) in terms of the deepest stratum. The only relevant character here is the "punctual" locus F_n. It will be enough to know that

$$\chi(F_j,\nu_j) = (-1)^j\chi(F_j), \tag{4.4.3}$$

which follows from [9, Cor. 3.5]. Note that here $\chi(F_j) = \chi(M_j)$ counts the number of fixed points of the torus action we have recalled in Section 4.2.1.

The Behrend function

Recall from [5] that any complex scheme Z carries a canonical constructible function $\nu_Z : Z \to \mathbb{Z}$. This is the "Behrend function" of Definition 1.1.1, which already made its appearance in the course of this chapter. In Definition 1.1.2 we recalled the weighted Euler characteristic

$$\tilde{\chi}(Z) = \chi(Z, \nu_Z) = \sum_{k \in \mathbb{Z}} k\chi(\nu_Z^{-1}(k)).$$

Given a morphism $f : Z \to X$, Behrend also considered the *relative* weighted Euler characteristic

$$\tilde{\chi}(Z, X) = \chi(Z, f^* \nu_X).$$

We now list its main properties following [5, Prop. 1.8]. First of all, it is clear that $\tilde{\chi}(Z) = \tilde{\chi}(Z, Z)$ through the identity map on Z.

(B1) If $Z = Z_1 \amalg Z_2$ for $Z_i \subset Z$ locally closed, then

$$\tilde{\chi}(Z, X) = \tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X).$$

(B2) Given two morphisms $Z_i \to X_i$, $i = 1, 2$, we have

$$\tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2) = \tilde{\chi}(Z_1, X_1) \cdot \tilde{\chi}(Z_2, X_2).$$

(B3) Given a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

with $X \to Y$ smooth and $Z \to W$ finite étale of degree d, we have

$$\tilde{\chi}(Z, X) = d(-1)^{\dim X/Y} \tilde{\chi}(W, Y).$$

(B4) This is a special case of (B3): if $X \to Y$ is étale (for instance, an open immersion), then $\tilde{\chi}(Z, X) = \tilde{\chi}(Z, Y)$.

4.4.2 The computation

We can start the proof of Theorem 4.4.1. Let us shorten $Y_0 = Y \setminus C$ for convenience. After fixing a partition $\alpha \vdash j$, let

$$V_\alpha \subset \prod_i Q_C^{\alpha_i}$$

denote the open subscheme consisting of tuples $(F_1, \ldots, F_{r_\alpha})$ of sheaves with pairwise disjoint support. According to Corollary 4.3.9, we can use the étale cover $\amalg_i Y \to Y$ to produce an étale morphism

$$f_\alpha : V_\alpha \to Q_C^j.$$

It is given on points by taking the "union" of the 0-dimensional supports of the sheaves F_i. Letting U_α be the image of f_α, we can form the diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d, "\text{Galois}"'] \arrow[dr, phantom, "\square"] & V_\alpha \arrow[r, hook, "\text{open}"] \arrow[d, "f_\alpha"] & \prod_i Q_C^{\alpha_i} \\ W_C^\alpha \arrow[r, hook] & U_\alpha \arrow[r, hook, "\text{open}"] & Q_C^j \end{tikzcd}$$

where the Cartesian square defines the scheme Z_α. The morphism on the left is Galois with Galois group G_α, the automorphism group of the partition α. It is easy to see that in fact

$$Z_\alpha = \prod_i W_C^{(\alpha_i)} \setminus \Delta$$

also fits in the Cartesian square

$$\begin{tikzcd} Z_\alpha \arrow[r, hook, "\text{open}"] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i W_C^{(\alpha_i)} \arrow[d, "\pi_\alpha"] \\ C^{r_\alpha} \setminus \Delta \arrow[r, hook, "\text{open}"] & C^{r_\alpha} \end{tikzcd} \tag{4.4.4}$$

where $W_C^{(\alpha_i)} \subset Q_C^{\alpha_i}$ is the deep stratum, Δ denotes the "big diagonal" (where at least two entries are equal), and the vertical map π_α is the product of the fibrations $\pi_C : W_C^{(\alpha_i)} \to C$, for $i = 1, \dots, r_\alpha$.

We need two identities before we can finish the computation.

First identity. We have

$$\chi(W_C^\alpha) = |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}). \tag{4.4.5}$$

Indeed, for each α, the map

$$\pi_\alpha : Z_\alpha \to C^{r_\alpha} \setminus \Delta$$

appearing in (4.4.4) is Zariski locally trivial with fiber $\prod_i F_{\alpha_i}$ by Corollary 4.3.11. Formula (4.4.5) follows since W_C^α is the free quotient Z_α / G_α.

Second identity. We have

$$\tilde{\chi}\Big(Z_\alpha, \prod_i Q_C^{\alpha_i}\Big) = \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}). \tag{4.4.6}$$

Indeed, by Corollary 4.3.11, we can find a Zariski open cover $\{B_s\}_s$ of $C^{r_\alpha} \setminus \Delta$ such that

$$(\pi_\alpha^{-1} B_s, \nu) \cong (B_s, 1_{B_s}) \times \Big(\prod_i F_{\alpha_i}, \prod_i \nu_{\alpha_i}\Big).$$

In the left hand side, ν denotes the Behrend function restricted from $\prod_i Q_C^{\alpha_i}$. We can refine this to a locally closed stratification $\amalg_\ell U_\ell = C^{r_\alpha} \setminus \Delta$ such that each U_ℓ is contained in some B_s. Therefore,

$$\begin{aligned}
\tilde{\chi}\Big(Z_\alpha, \prod_i Q_C^{\alpha_i}\Big) &= \sum_\ell \tilde{\chi}\Big(\pi_\alpha^{-1} U_\ell, \prod_i Q_C^{\alpha_i}\Big) && \text{by (B1)}\\
&= \sum_\ell \chi\Big(U_\ell \times \prod_i F_{\alpha_i}, 1_{U_\ell} \times \prod_i \nu_{\alpha_i}\Big) \\
&= \sum_\ell \chi(U_\ell, 1_{U_\ell}) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}) && \text{by (B2)}\\
&= \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}),
\end{aligned}$$

and (4.4.6) is proved.

Note that combining (4.4.1) and (4.4.5) we get

$$\chi(Q_C^n) = \sum_{j,\alpha} \chi(\mathrm{Hilb}^{n-j} Y_0) \cdot |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}). \tag{4.4.7}$$

We now have all the tools to finish the computation. Let us fix j and a partition $\alpha \vdash j$. We define

$$D_\alpha \subset \mathrm{Hilb}^{n-j} Y \times \prod_i Q_C^{\alpha_i}$$

to be the set of tuples $(Z_0, F_1, \ldots, F_{r_\alpha})$ such that $(F_1, \ldots, F_{r_\alpha}) \in V_\alpha$ and the support of Z_0 does not meet the support of any F_i. Then D_α is an open subscheme. The Galois cover $1 \times f_\alpha : \mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha \to \mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha$ extends to an étale map $D_\alpha \to Q_C^n$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha \arrow[r, hook] \arrow[d, "1\times f_\alpha"'] & D_\alpha \arrow[d, "\text{ét}"] \\
\mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha \arrow[r, hook] & Q_C^n.
\end{tikzcd} \tag{4.4.8}$$

Therefore we can start computing $\tilde{\chi}(Q_C^n) = \chi(Q_C^n, \nu_{Q_C^n})$ as follows:

$$\begin{aligned}
\tilde{\chi}(Q_C^n) &= \sum_{j,\alpha} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha, Q_C^n) && \text{by (B1) applied to (4.4.1)}\\
&= \sum_{j,\alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha, D_\alpha) && \text{by (B3) applied to (4.4.8)}\\
&= \sum_{j,\alpha} |G_\alpha|^{-1} \tilde{\chi}\Big(\mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha, \mathrm{Hilb}^{n-j} Y \times \prod_i Q_C^{\alpha_i}\Big) && \text{by (B4)}\\
&= \sum_{j,\alpha} |G_\alpha|^{-1} \tilde{\chi}\Big(\mathrm{Hilb}^{n-j} Y_0, \mathrm{Hilb}^{n-j} Y\Big) \cdot \tilde{\chi}\Big(Z_\alpha, \prod_i Q_C^{\alpha_i}\Big) && \text{by (B2)}\\
&= \sum_{j,\alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0) \cdot \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}) && \text{by (B4) and (4.4.6)}\\
&= (-1)^n \sum_{j,\alpha} \chi(\mathrm{Hilb}^{n-j} Y_0) \cdot |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}) && \text{by (4.4.2) and (4.4.3)}\\
&= (-1)^n \chi(Q_C^n) && \text{by (4.4.7).}
\end{aligned}$$

This completes the proof of Theorem 4.4.1.

Question 4.4.1. It would be nice to know whether the Behrend function on $M_n = \mathrm{Quot}_n(\mathscr{I}_L)$ is the constant sign $(-1)^n$. As far as we know, this is still open even when the curve is absent, namely for $\mathrm{Hilb}^n \mathbb{A}^3$.

4.5 Ideals, pairs and quotients

In this section we give some applications of the formula

$$\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n).$$

We show that the DT/PT correspondence holds for the contribution of a smooth *rigid* curve in a projective Calabi–Yau threefold. We discuss, at a conjectural level, the case of an arbitrary smooth curve.

4.5.1 Local contributions

We fix a smooth projective threefold Y and a Cohen-Macaulay curve $C \subset Y$ of arithmetic genus $g = 1 - \chi(\mathscr{O}_C)$, embedded in class $\beta \in H_2(Y, \mathbb{Z})$. We will use the Quot scheme to endow the closed subset

$$\left\{ Z \subset Y \,\middle|\, C \subset Z, \chi(\mathscr{I}_C / \mathscr{I}_Z) = n \right\} \subset I_{1-g+n}(Y, \beta)$$

with a natural scheme structure.

LEMMA 4.5.1. *There is a closed immersion* $\iota : Q_C^n \to I_{1-g+n}(Y, \beta)$.

PROOF. Let $\mathscr{I}_{C\times T} \twoheadrightarrow \mathscr{F}$ be a flat family of quotients parametrized by a scheme T. Letting $Z \subset Y \times T$ be the subscheme defined by the kernel of the surjection, we get an exact sequence

$$0 \to \mathscr{F} \to \mathscr{O}_Z \to \mathscr{O}_{C\times T} \to 0.$$

The middle term is flat over T, therefore it determines a point in the Hilbert scheme of Y. The discrete invariants β and $\chi = 1 - g + n$ are the right ones, as one can see by restricting the above short exact sequence to closed points of T. Therefore we get a morphism

$$\iota : Q_C^n \to I_{1-g+n}(Y, \beta).$$

The correspondence at the level of functor of points is injective, and the morphism is proper (since the Quot scheme is proper, as Y is projective). Moreover ι is injective at the level of tangent spaces; indeed, the tangent map

$$\mathrm{Hom}(\mathscr{I}_Z, \mathscr{F}) \to \mathrm{Hom}(\mathscr{I}_Z, \mathscr{O}_Z)$$

obtained by applying $\mathrm{Hom}(\mathscr{I}_Z, -)$ to the above exact sequence, is injective for all $[\mathscr{F}] \in Q_C^n$. But a proper morphism that is injective on points and on tangent spaces is a closed immersion. $\square$

Definition 4.5.2. We define

$$I_n(Y,C) \subset I_{1-g+n}(Y,\beta) \tag{4.5.1}$$

to be the scheme-theoretic image of $\iota : Q_C^n \to I_{1-g+n}(Y,\beta)$. ◊

Remark 4.5.3. The closed subset $|I_n(Y,C)| \subset I_{1-g+n}(Y,\beta)$ also has a scheme structure induced by GIT wall-crossing [75]. Another scheme structure is defined in the recent paper [18]. See in particular Definition 4, where the notation used is $\mathrm{Hilb}^n(Y,C)$. We believe both these scheme structures agree with the one of our Definition 4.5.2, in which case they describe schemes isomorphic to Q_C^n. ◊

Assume Y is a projective *Calabi–Yau* threefold. By the main result of [5], the degree β curve counting invariants

$$\mathsf{DT}_{m,\beta} = \int_{[I_m(Y,\beta)]^{\mathrm{vir}}} 1, \qquad \mathsf{PT}_{m,\beta} = \int_{[P_m(Y,\beta)]^{\mathrm{vir}}} 1$$

can be computed as weighted Euler characteristics of the corresponding moduli spaces, since the obstruction theories defining the virtual cycles are symmetric. One can define the contribution of C to the above invariants as

$$\mathsf{DT}_{n,C} = \chi(I_n(Y,C),\nu_I), \qquad \mathsf{PT}_{n,C} = \chi(P_n(Y,C),\nu_P). \tag{4.5.2}$$

Here we have set $I = I_{1-g+n}(Y,\beta)$ and $P = P_{1-g+n}(Y,\beta)$. The subscheme $P_n(Y,C) \subset P$ consists of stable pairs with Cohen-Macaulay support equal to C. Note that these integers remember how C sits inside Y, since the weight is the Behrend function coming from the full moduli space.

An immediate consequence of Theorem 4.4.1 is a formula for the DT contribution of a smooth rigid curve.

THEOREM 4.5.4. *Let Y be a projective Calabi–Yau threefold, $C \subset Y$ a smooth rigid curve. Then*

$$\mathsf{DT}_{n,C} = (-1)^n \chi(I_n(Y,C)).$$

PROOF. The inclusion (4.5.1) is both open and closed thanks to the infinitesimal isolation of C. Then $\nu_I|_{I_n(Y,C)} = \nu_{I_n(Y,C)}$, thus

$$\mathsf{DT}_{n,C} = \tilde{\chi}(I_n(Y,C)) = (-1)^n \chi(I_n(Y,C)),$$

as claimed. □

Remark 4.5.5. In the rigid case, $\mathsf{DT}_{n,C}$ is a DT invariant in the classical sense, namely it is the degree of the virtual class $[I_n(Y,C)]^{\mathrm{vir}}$ obtained by restricting the one on $I_{1-g+n}(Y,\beta)$. ◊

Theorem 4.5.4 can be seen as an instance of the following more general result, which is also a direct consequence of Theorem 4.4.1.

Proposition 4.5.6. *Let Y be a smooth projective threefold. If $C \subset Y$ is a smooth curve of genus g, then*

$$\sum_{n\geq 0} \tilde{\chi}(I_n(Y,C))q^n = M(-q)^{\chi(Y)}(1+q)^{2g-2}. \tag{4.5.3}$$

Proof. For any smooth threefold X we have Cheah's formula [22]

$$\sum_{n\geq 0} \chi(\mathrm{Hilb}^n X)q^n = M(q)^{\chi(X)}.$$

On the other hand, for every partition α of n, written in the form

$$\alpha = (1^{\alpha_1}\cdots i^{\alpha_i}\cdots \ell^{\alpha_\ell}),$$

we have a Zariski locally trivial fibration

$$W_C^\alpha \to \mathrm{Sym}_\alpha^n C$$

with fibre $\prod_i F_i^{\alpha_i}$. Therefore

$$\chi(W_C^n) = \sum_{\alpha\vdash n} \chi(\mathrm{Sym}_\alpha^n C)\cdot \prod_i \chi(F_i)^{\alpha_i}$$

so the natural power structure on $\mathbb{Z}$ recalled in (2.2.1) yields

$$\sum_{n\geq 0} \chi(W_C^n)q^n = \left(\sum_{n\geq 0}\chi(F_n)q^n\right)^{\chi(C)}.$$

Applying Cheah's formula to $X = Y\setminus C$, we compute

$$\begin{aligned}
\sum_{n\geq 0} \chi(I_n(Y,C))q^n &= M(q)^{\chi(Y\setminus C)}\cdot \left(\sum_{n\geq 0}\chi(F_n)q^n\right)^{\chi(C)} && \text{by (4.4.1)}\\
&= M(q)^{\chi(Y\setminus C)}\cdot \left(\sum_{n\geq 0}\chi(M_n)q^n\right)^{\chi(C)} && \text{as } \chi(F_n)=\chi(M_n)\\
&= M(q)^{\chi(Y\setminus C)}\cdot \left(\frac{M(q)}{1-q}\right)^{\chi(C)} && \text{by (4.2.2)}\\
&= M(q)^{\chi(Y)}(1-q)^{2g-2}.
\end{aligned}$$

The claimed formula follows by Theorem 4.4.1. □

Remark 4.5.7. Formula (4.5.3) can be rewritten as

$$\sum_{n\geq 0} \tilde{\chi}(I_n(Y,C))q^n = M(-q)^{\chi(Y)}\sum_{n\geq 0}\tilde{\chi}(P_n(Y,C))q^n. \tag{4.5.4}$$

Indeed $P_n(Y,C) = \mathrm{Sym}^n C$ is smooth of dimension n, thus $\tilde{\chi} = (-1)^n\chi$. The latter identity can be seen as the ν-weighted version of the "local" wall-crossing formula between ideals and stable pairs, which was already established for a single Cohen-Macaulay curve at the level of Euler characteristics [75, Thm. 1.5]. In other words, (4.5.4) is precisely what happens to the Stoppa–Thomas identity

$$\sum_{n\geq 0} \chi(I_n(Y,C))q^n = M(q)^{\chi(Y)}\sum_{n\geq 0}\chi(P_n(Y,C))q^n$$

when we replace q by $-q$. ♢

4.5.2 DT/PT wall-crossing at a single curve

Let C be a smooth curve of genus g, embedded in class β in a smooth projective Calabi–Yau threefold Y. Let us define the generating series

$$\mathsf{DT}_C(q) = \sum_{n\geq 0} \mathsf{DT}_{n,C}\, q^n$$
$$\mathsf{PT}_C(q) = \sum_{n\geq 0} \mathsf{PT}_{n,C}\, q^n$$

encoding the local contributions defined in (4.5.2). The stable pair side has already been computed [63, Lemma 3.4]. The result is

$$\mathsf{PT}_C(q) = n_{g,C}\cdot (1+q)^{2g-2}, \tag{4.5.5}$$

where $n_{g,C}$ is the g-th BPS number of C. For instance, if C is rigid, then $n_{g,C} = 1$ and thanks to Theorem 4.5.4 we see that (4.5.3) can be rewritten as

$$\mathsf{DT}_C(q) = M(-q)^{\chi(Y)}\cdot \mathsf{PT}_C(q).$$

This formula can be seen as a "local DT/PT correspondence", or local wall-crossing formula at C. We next prove that such formula, for arbitrary C, is equivalent to the following conjecture.

Conjecture 1. Let C be a smooth curve in a projective Calabi–Yau threefold Y. Let $\mathcal{I} = I_{1-g}(Y,\beta)$ be the Hilbert scheme where the ideal sheaf of C lives as a point. Then, for all n, one has

$$\mathsf{DT}_{n,C} = \nu_{\mathcal{I}}(\mathscr{I}_C)\cdot \tilde{\chi}(I_n(Y,C)). \qquad ♣$$

Remark 4.5.8. An equivalent formula has been conjectured by Bryan and Kool in their recent paper [18]. See Conjecture 18 in *loc. cit.* for the precise (more general) setting. ◇

THEOREM 4.5.9. *Let Y be a projective Calabi–Yau threefold, $C \subset Y$ a smooth curve. Then Conjecture 1 is equivalent to the wall-crossing identity*

$$\mathsf{DT}_C(q) = M(-q)^{\chi(Y)}\cdot \mathsf{PT}_C(q).$$

PROOF. Combining (4.5.5) with (4.5.3), we see that the right hand side of the formula equals

$$n_{g,C}\cdot \sum_{n\geq 0} \tilde{\chi}(I_n(Y,C))q^n.$$

Therefore the DT/PT correspondence holds at C if and only if

$$\mathsf{DT}_{n,C} = n_{g,C}\cdot \tilde{\chi}(I_n(Y,C)).$$

We are then left with proving that $\nu_{\mathcal{I}}(\mathscr{I}_C) = n_{g,C}$. Recall that the moduli space of ideal sheaves is isomorphic to the moduli space of stable pairs along the *open* subschemes parametrizing *pure* curves. Moreover, the map $\phi : P_{1-g}(Y,\beta) \to$

$\mathcal{M}$ to the moduli space of stable pure sheaves considered in [63], defined by forgetting the section of a stable pair, satisfies the relation

$$\nu_{P_{1-g}(Y,\beta)} = (-1)^g \phi^* \nu_{\mathcal{M}}$$

by [63, Thm. 4]. Hence

$$\begin{aligned}\nu_{\mathcal{I}}(\mathscr{I}_C) &= \nu_{\mathcal{I}^{\mathrm{pur}}}(\mathscr{I}_C)\\ &= \nu_{P_{1-g}(Y,\beta)}([\mathscr{O}_Y \twoheadrightarrow \mathscr{O}_C])\\ &= (-1)^g \nu_{\mathcal{M}}(\mathscr{O}_C)\\ &= n_{g,C}\end{aligned}$$

where the last equality is [63, Prop. 3.6]. □

Remark 4.5.10. Thanks to the identity $\nu_{\mathcal{I}}(\mathscr{I}_C) = n_{g,C}$, proved in the course of Theorem 4.5.9, Conjecture 1 can be rephrased as

$$\mathsf{DT}_{n,C} = \nu_P\big|_{P_n(Y,C)} \cdot \chi(I_n(Y,C)),$$

where $\nu_P|_{P_n(Y,C)}$ is the constant $(-1)^n \cdot n_{g,C} = (-1)^{n-g}\,\nu_{\mathcal{M}}(\mathscr{O}_C)$. In particular the conjecture says that the DT and PT contributions of C differ from the Euler characteristic of the corresponding moduli space by the *same* constant. ◊

We end this chapter with some speculations, indicating plausibility reasons why Conjecture 1 should hold true.

Suppose we were able to show that, given a point $\mathscr{I}_Z \in I_n(Y,C) \subset I$, a formal neighborhood of $\mathscr{I}_Z$ in I is isomorphic to a product

$$U \times V,$$

where U is a formal neighborhood of $\mathscr{I}_C$ in $\mathcal{I}$ and V is a formal neighborhood of $\mathscr{I}_Z$ in $I_n(Y,C)$. Then, since the Behrend function value $\nu(P)$ only depends on a formal neighborhood of P [40], this would immediately lead to the Behrend function identity

$$\nu_I\big|_{I_n(Y,C)} = \nu_{\mathcal{I}}(\mathscr{I}_C) \cdot \nu_{I_n(Y,C)}, \tag{4.5.6}$$

from which Conjecture 1 follows after integration. One reason to believe in a product decomposition as above is the following. At least when the maximal purely 1-dimensional part $C \subset Z$ is smooth, one may expect to be able to "separate" infinitesimal deformations of C (the factor U) from those deformations of Z that keep C fixed (the factor V in the Quot scheme). This decomposition is manifestly false when C acquires a singularity, and we do not know of any counterexample in the smooth case.

5 THE DT/PT CORRESPONDENCE FOR SMOOTH CURVES

5.1 Introduction

This chapter is essentially the content of [69]. The purpose is to prove Conjecture 1 (see p. 58), so far only established for rigid curves. The main result will then be the following.

THEOREM 5.1.1. *Let Y be a smooth, projective Calabi–Yau threefold, $C \subset Y$ a smooth curve. Then the DT/PT correspondence holds for C,*

$$\mathsf{DT}_C(q) = \mathsf{DT}_0(Y,q) \cdot \mathsf{PT}_C(q). \tag{5.1.1}$$

Here $\mathsf{DT}_0(Y,q)$ is the MacMahon factor $M(-q)^{\chi(Y)}$.

In fact, the conclusion of the theorem holds for all Cohen–Macaulay curves, by recent work of Oberdieck [60]. While he works with motivic Hall algebras, our method is geometric, combining results from the previous chapter with a local study of the Hilbert–Chow morphism.

Conventions. The Calabi–Yau condition, as usual, is simply the existence of a trivialization of the canonical line bundle. The *Chow functor* of a projective variety Y is the one constructed by D. Rydh, as well as the Hilbert–Chow morphism $\mathrm{Hilb}_r(Y) \to \mathrm{Chow}_r(Y)$. We refer to [70] for all details regarding these constructions.

5.2 The DT/PT correspondence

In this section we outline our strategy to deduce Theorem 5.1.1.

Let Y be a smooth projective variety, not necessarily Calabi–Yau. We consider the Hilbert–Chow morphism

$$\mathrm{Hilb}_1(Y) \to \mathrm{Chow}_1(Y) \tag{5.2.1}$$

constructed in [70], sending a 1-dimensional subscheme of Y to its fundamental cycle. We recall its definition in Section 5.3.1. Let $I_m(Y,\beta) \subset \mathrm{Hilb}_1(Y)$ be the component parametrizing subschemes $Z \subset Y$ such that

$$\chi(\mathscr{O}_Z) = m \in \mathbb{Z}, \qquad [Z] = \beta \in H_2(Y,\mathbb{Z}).$$

Similarly, we let $\mathrm{Chow}_1(Y,\beta) \subset \mathrm{Chow}_1(Y)$ be the component parametrizing 1-cycles of degree β. Then (5.2.1) restricts to a morphism

$$\mathsf{h}_m : I_m(Y,\beta) \to \mathrm{Chow}_1(Y,\beta).$$

Definition 5.2.1. Fix an integer $n \geq 0$. For a Cohen–Macaulay curve $C \subset Y$ of arithmetic genus g embedded in class β, we let

$$I_n(Y,C) \subset I_{1-g+n}(Y,\beta)$$

denote the scheme-theoretic fibre of h_{1-g+n}, over the cycle of C. ◊

Remark 5.2.2. We will use that the natural transformation (5.2.1) is an isomorphism around normal schemes, at least in characteristic zero [70, Cor. 12.9]. Thus, for a smooth curve $C \subset Y$, we will identify Chow with Hilb locally around the cycle $[C] \in \mathrm{Chow}_1(Y)$ and the ideal sheaf $\mathscr{I}_C \in \mathrm{Hilb}_1(Y)$. For this reason, we will not need the representability of the global Chow functor in what follows, as around the point $[C] \in \mathrm{Chow}_1(Y,\beta)$ we can work with the ideal sheaf $\mathscr{I}_C \in I_{1-g}(Y,\beta)$ instead. ◊

Consider the Quot scheme

$$\mathrm{Quot}_n(\mathscr{I}_C)$$

parametrizing quotients of length n of the ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$. We proved in Lemma 4.5.1 that the association $[\theta : \mathscr{I}_C \twoheadrightarrow \mathscr{E}] \mapsto \ker\theta$ defines a closed immersion

$$\mathrm{Quot}_n(\mathscr{I}_C) \hookrightarrow I_{1-g+n}(Y,\beta). \tag{5.2.2}$$

Recall that for a scheme S, an S-valued point of the Quot scheme is a flat quotient $\mathscr{E} = \mathscr{I}_{C\times S}/\mathscr{I}_Z$, and in the short exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_Z \to \mathscr{O}_{C\times S} \to 0$$

over $Y \times S$, the middle term is S-flat, so Z defines an S-point of $I_{1-g+n}(Y,\beta)$. The S-valued points of the image of (5.2.2) consist precisely of those flat families $Z \subset Y \times S \to S$ such that Z contains $C \times S$ as a closed subscheme. This will be used implicitly in the proof of Theorem 5.2.3.

The schemes $I_n(Y,C)$ and $\mathrm{Quot}_n(\mathscr{I}_C)$ have the same $\mathbb{C}$-valued points: they both parametrize subschemes $Z \subset Y$ consisting of C together with "n points", possibly embedded. The first step towards Theorem 5.1.1 is the following result, whose proof is postponed to the next section.

THEOREM 5.2.3. *Let Y be a smooth projective variety, $C \subset Y$ a smooth curve of genus g. Then $I_n(Y,C) = \mathrm{Quot}_n(\mathscr{I}_C)$ as subschemes of $I_{1-g+n}(Y,\beta)$.*

As an application of Theorem 5.2.3, in Section 5.4 we compute the reduced Donaldson–Thomas theory of a general Abel–Jacobi curve of genus 3.

To proceed towards Theorem 5.1.1, we need to examine the local structure of the Hilbert scheme around subschemes $Z \subset Y$ whose maximal purely 1-dimensional subscheme $C \subset Z$ is smooth. The result, given below, will be proven in the next section.

THEOREM 5.2.4. *Let Y be a smooth projective variety, $C \subset Y$ a smooth curve of genus g. Then, locally analytically around $I_n(Y,C)$, the Hilbert scheme $I_{1-g+n}(Y,\beta)$ is isomorphic to $I_n(Y,C) \times \mathrm{Chow}_1(Y,\beta)$.*

Roughly speaking, this means that the Hilbert–Chow morphism, locally about the cycle

$$[C] \in \mathrm{Chow}_1(Y,\beta),$$

behaves like a fibration with typical fibre $I_n(Y,C)$. To obtain this, we first identify Chow with Hilb locally around C, cf. Remark 5.2.2. We then need to trivialize the universal curve $\mathscr{C} \to \mathrm{Hilb}$, which can be done since smooth maps are analytically locally trivial (on the source). However, even if we had $\mathscr{C} = C \times \mathrm{Hilb}$, we would not be done: the fibre of Hilbert–Chow (which is the Quot scheme by Theorem 5.2.3) depends on the embedding of the curve into Y, not just on the abstract curve. So to prove Theorem 5.2.4 we need to trivialize (locally) the embedding of the universal curve into $Y \times \mathrm{Hilb}$. This is taken care of by a local-analytic version of the tubular neighborhood theorem. After this step, Theorem 5.2.4 follows easily.

Granting Theorems 5.2.3 and 5.2.4, we can prove the DT/PT correspondence for smooth curves. So now we assume C is a smooth curve embedded in class β in a smooth, projective Calabi–Yau threefold Y.

PROOF OF THEOREM 5.1.1. By [70, Cor. 12.9], the Hilbert–Chow morphism

$$\mathsf{h}_{1-g} : I_{1-g}(Y,\beta) \to \mathrm{Chow}_1(Y,\beta)$$

is (in characteristic zero) an isomorphism over the locus of normal schemes. Under this local identification, the cycle $[C]$ corresponds to the ideal sheaf $\mathscr{I}_C$. We let $\nu(\mathscr{I}_C)$ be the value of the Behrend function on $I_{1-g}(Y,\beta)$ at the point corresponding to $\mathscr{I}_C$. Since the Behrend function can be computed locally analytically [5, Prop. 4.22], Theorem 5.2.4 implies the identity

$$\nu_I\big|_{I_n(Y,C)} = \nu(\mathscr{I}_C)\cdot \nu_{I_n(Y,C)},$$

where ν_I is the Behrend function of $I = I_{1-g+n}(Y,\beta)$. After integration, we find

$$\mathsf{DT}_{n,C} = \nu(\mathscr{I}_C)\cdot \tilde{\chi}(I_n(Y,C)),$$

where $\tilde{\chi}(I_n(Y,C))$, by Theorem 5.2.3, agrees with the weighted Euler characteristic of the Quot scheme $\mathrm{Quot}_n(\mathscr{I}_C)$. But we proved in Theorem 4.5.9 that the relation

$$\mathsf{DT}_{n,C} = \nu(\mathscr{I}_C)\cdot \tilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C))$$

is equivalent to the C-local DT/PT correspondence expressed in (5.1.1), so the theorem follows. □

As observed in Section 4.5, the local DT/PT correspondence says that the local invariants are determined by the topological Euler characteristic of the corresponding moduli space, along with the BPS number of the fixed smooth curve $C \subset Y$. The latter can be computed as

$$n_{g,C} = \nu(\mathscr{I}_C).$$

For any integer $n \geq 0$, the formulas are

$$\begin{aligned}\mathsf{DT}_{n,C} &= n_{g,C} \cdot (-1)^n \chi(I_n(Y,C)),\\ \mathsf{PT}_{n,C} &= n_{g,C} \cdot (-1)^n \chi(P_n(Y,C)).\end{aligned}$$

In particular, the local invariants differ by the Euler characteristic of the corresponding moduli space by the *same* constant.

5.3 Proofs

It remains to prove Theorems 5.2.3 and 5.2.4. For Theorem 5.2.3, we need to review some definitions and results from [70].

5.3.1 The fibre of Hilbert–Chow

Rydh has developed a powerful theory of *relative cycles* and has defined a Hilbert–Chow morphism

$$\mathrm{Hilb}_r(X/S) \to \mathrm{Chow}_r(X/S) \tag{5.3.1}$$

for every algebraic space X locally of finite type over an arbitrary scheme S. For us X is always a scheme, projective over S.

We quickly recall the definition of (5.3.1). First of all, the Hilbert scheme $\mathrm{Hilb}_r(X/S)$ parametrizes S-subschemes of X that are proper and of dimension r over S, but not necessarily equidimensional, while the Chow functor $\mathrm{Chow}_r(X/S)$ classifies *equidimensional*, proper relative cycles of dimension r. We refer to [70, Def. 4.2] for the definition of *relative cycles* on X/S. Cycles have a (not necessarily equidimensional) support, which is a locally closed subset $Z \subset X$. Rydh shows [70, Prop. 4.5] that if α is a relative cycle on $f: X \to S$ with support Z, then, for every $r \geq 0$, on the same family there is a unique *equidimensional* relative cycle α_r with support

$$Z_r = \left\{ x \in Z \,\middle|\, \dim_x Z_{f(x)} = r \right\} \subset Z.$$

Cycles are called equidimensional when their support is equidimensional over the base. The essential tool for the definition of (5.3.1) is the *norm family*, defined by the following result.

THEOREM 5.3.1 ([70, Thm. 7.14]). *Let $X \to S$ be a locally finitely presented morphism, $\mathcal{F}$ a finitely presented $\mathcal{O}_X$-module which is flat over S. Then there is a canonical relative cycle $\mathcal{N}_{\mathcal{F}}$ on X/S, with support equal to* Supp $\mathcal{F}$. *This construction commutes with arbitraty base change. When $Z \subset X$ is a subscheme which is flat and of finite presentation over S, we write $\mathcal{N}_Z = \mathcal{N}_{\mathcal{O}_Z}$.*

The Hilbert–Chow functor (5.3.1) is defined by $Z \mapsto (\mathcal{N}_Z)_r$.

Even though we do not recall here the full definition of relative cycle, the main idea is the following. For a locally closed subset $Z \subset X$, Rydh defines a *projection of X/S adapted to Z* to be a commutative diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d] & X \arrow[dd] \\ B \arrow[d] & \\ T \arrow[r, "g"] & S \end{tikzcd} \tag{5.3.2}$$

where $U \to X \times_S T$ is étale, $B \to T$ is smooth and $p^{-1}(Z) \to B$ is finite. A relative cycle α on X/S with support $Z \subset X$ is the datum, for every projection adapted to Z, of a proper family of *zero*-cycles on U/B, which Rydh defines as a morphism

$$\alpha_{U/B/T} : B \to \Gamma^\star(U/B)$$

to the scheme of divided powers. We refer to [70, Def. 4.2] for the additional compatibility conditions that these data should satisfy.

Let now $\mathcal{F}$ be a flat family of coherent sheaves on X/S. If $\mathrm{p} = (U, B, T, p, g)$ denotes a projection of X/S adapted to $\operatorname{Supp} \mathcal{F} \subset X$ as in (5.3.2), then the *zero*-cycle defining the norm family $\mathcal{N}_{\mathcal{F}}$ at p is

$$(\mathcal{N}_{\mathcal{F}})_{U/B/T} = \mathcal{N}_{p^*\mathcal{F}/B},$$

constructed in [70, Cor. 7.9]. For us $\mathcal{F}$ will always be a structure sheaf, so it will be easy to compare these zero-cycles.

If $Z \subset X$ is a subscheme that is smooth over S, then the norm family $\mathcal{N}_Z$ is an example of a *smooth* relative cycle, cf. [70, Def. 8.11]. The next result states an equivalence, in characteristic zero, between smooth relative cycles and subschemes smooth over the base.

THEOREM 5.3.2 ([70, Thm. 9.8]). *If S is of characteristic zero, then for every smooth relative cycle α on X/S there is a unique subscheme $Z \subset X$, smooth over S, such that $\alpha = \mathcal{N}_Z$.*

We can now prove Theorem 5.2.3. We fix Y to be a smooth projective variety, $C \subset Y$ a smooth curve of genus g in class β, and we denote by $I_n(Y,C)$ the fibre over $[C]$ of the Hilbert–Chow morphism

$$I_{1-g+n}(Y,\beta) \to \mathrm{Chow}_1(Y,\beta),$$

as in Definition 5.2.1.

PROOF OF THEOREM 5.2.3. We need to show the equality

$$I_n(Y,C) = \mathrm{Quot}_n(\mathscr{I}_C)$$

as subschemes of $I_{1-g+n}(Y,\beta)$. Let S be a scheme over $\mathbb{C}$, and set $X = Y \times S$. Then a family

$$Z \subset X \to S$$

in the Hilbert scheme is an S-valued point of $I_n(Y,C)$ when $(\mathcal{N}_Z)_1 = \mathcal{N}_{C\times S}$. The closed immersion (5.2.2) from the Quot scheme to the Hilbert scheme factors through $I_n(Y,C)$. Indeed, any S-point $\mathscr{I}_{C\times S} \twoheadrightarrow \mathscr{I}_{C\times S}/\mathscr{I}_Z$ of the Quot scheme gives a closed immersion $C\times S \hookrightarrow Z$ whose relative ideal is of dimension zero over S, thus we have $(\mathcal{N}_Z)_1 = (\mathcal{N}_{C\times S})_1 = \mathcal{N}_{C\times S}$, where in the second equality we used that $\mathcal{N}_{C\times S}$ is equidimensional of dimension one over S. So we obtain a closed immersion

$$\iota : \mathrm{Quot}_n(\mathscr{I}_C) \hookrightarrow I_n(Y,C).$$

For every scheme S, we have an injective map of sets

$$\iota(S) : \mathrm{Quot}_n(\mathscr{I}_C)(S) \hookrightarrow I_n(Y,C)(S),$$

and since $\iota(\operatorname{Spec}\mathbb{C})$ is a bijection, so far ι is just a bijective closed immersion. We need to show $\iota(S)$ is onto, and for the moment we deal with the case where S is a fat point. In other words, assume S is the spectrum of a local artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Let $Z\subset X\to S$ be an S-valued point of $I_n(Y,C)$. Consider the finite subscheme $F\subset Y\subset X$ given by the support of $\mathscr{I}_C/\mathscr{I}_{Z_0}$, where Z_0 is the closed fibre of $Z\to S$. Form the open set $V = X\setminus F\subset X$. Then we have, as relative cycles on V/S,

$$(\mathcal{N}_Z)_1\big|_V = \mathcal{N}_{C\times S}\big|_V = \mathcal{N}_{(C\times S)\cap V}.$$

We claim the left hand side equals the relative cycle $\mathcal{N}_{Z\cap V}$. For sure, these two cycles have the same support, as $Z\cap V = Z_1\cap V$, and they are determined by the same set of projections; indeed, being equidimensional of dimension one, they are determined by (compatible data of) relative zero-cycles for every projection $\mathsf{p}_{V/S} = (U,B,T,p,g)$ such that B/T is smooth of relative dimension one. Let us focus on $(\mathcal{N}_Z)_1$ first. Here $r=1$ is the maximal relative dimension of a point in Z, so the zero-cycle corresponding to a projection $\mathsf{p}_{X/S}$ as in (5.3.2), and adapted to Z_1, is the same as the one defined by the norm family of Z (cf. the proof of [70, Prop. 4.5]), namely $\mathcal{N}_{p^*\mathscr{O}_Z/B}$. Now we restrict to the open subset $i: V\to X$. By definition of pullback, the zero-cycle attached to a projection $\mathsf{p}_{V/S}$ (adapted to $Z_1\cap V$) is the cycle corresponding to the projection $(U,B,T,i\circ p,g)$ for the full family Z/S, namely

$$\mathcal{N}_{(i\circ p)^*\mathscr{O}_Z/B} = \mathcal{N}_{p^*\mathscr{O}_{Z\cap V}/B}.$$

The latter is precisely the zero-cycle defining the norm family of $Z\cap V/S$ at the same projection $\mathsf{p}_{V/S}$, so the claim is proved,

$$\mathcal{N}_{Z\cap V} = (\mathcal{N}_Z)_1\big|_V.$$

By the equivalence between smooth cycles and smooth subschemes stated in Theorem 5.3.2, we conclude that $Z\cap V$ and $(C\times S)\cap V$ are the same (smooth) family over S. Moreover, the closure

$$\overline{(C\times S)\cap V}\subset Z$$

equals $C\times S$, because the open subscheme $(C\times S)\cap V\subset C\times S$ is fibrewise dense (intersecting with V is only deleting a finite number of points in the special fibre). We have thus reconstructed a closed immersion $C \times S \hookrightarrow Z$, giving a well-defined S-valued point of $\mathrm{Quot}_n(\mathscr{I}_C)$. So $\iota(S)$ is onto, and thus a bijection, whenever S is a fat point. This implies ι is étale, by a simple application of the formal criterion for étale maps. The theorem follows because we already know ι is a bijective closed immersion. □

5.3.2 Local triviality of Hilbert–Chow

In this section we prove Theorem 5.2.4. The main tool used in the proof is the following local analytic version of the tubular neighborhood theorem.

LEMMA 5.3.3. *Let S be a scheme, $j: X \to Y$ a closed immersion over S. Assume X and Y are both smooth over S, of relative dimension d and n respectively. Then j is locally analytically isomorphic to the standard linear embedding $\mathbb{C}^d \times S \to \mathbb{C}^n \times S$.*

PROOF. Let $x \in X$ and $y = j(x) \in Y$. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the ideal sheaf of X in Y. The relative smoothness of X, given that of Y, is characterized by the Jacobian criterion [11, Section 8.5], asserting that the short exact sequence

$$0 \to \mathscr{I}/\mathscr{I}^2 \to j^*\Omega_{Y/S} \to \Omega_{X/S} \to 0$$

is split locally around $x \in X$. According to *loc. cit.* this is also equivalent to the following: whenever we choose local sections $t_1, \ldots, t_n$ and $g_1, \ldots, g_N$ of $\mathscr{O}_{Y,y}$ such that $\mathrm{d}\,t_1, \ldots, \mathrm{d}\,t_n$ constitute a free generating system for $\Omega_{Y/S,y}$ and $g_1, \ldots, g_N$ generate $\mathscr{I}_y$, after a suitable relabeling we may assume $g_{d+1}, \ldots, g_n$ generate $\mathscr{I}$ about y and

$$\mathrm{d}\,t_1, \ldots, \mathrm{d}\,t_d, \mathrm{d}\,g_{d+1}, \ldots, \mathrm{d}\,g_n$$

generate $\Omega_{Y/S}$ locally around y. In particular, $f_i = t_i \circ j$, for $i = 1, \ldots, d$, define a local system of parameters at x. By this choice of local basis for $\Omega_{Y/S}$ around y, we can find open neighborhoods $x \in U \subset X$ and $y \in V \subset Y$ fitting in a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, "\text{ét}"'] & V \arrow[d, "\text{ét}"] \\ \mathbb{A}^d_S \arrow[r, hook] & \mathbb{A}^n_S \end{tikzcd}$$

where the vertical arrows are the étale maps defined by the local systems of parameters $(f_1, \ldots, f_d)$ and $(t_1, \ldots, t_d, g_{d+1}, \ldots, g_n)$ respectively, and the lower immersion is defined by sending $t_i \mapsto f_i$ for $i = 1, \ldots, d$ and $g_k \mapsto 0$. Using the analytic topology, the inverse function theorem allows us to translate the étale maps into local analytic isomorphisms, and the statement follows. □

Note that Lemma 5.3.3 does not hold globally. For a closed immersion $X \subset Y$ of smooth complex *projective* varieties, it is not true in general that one can find a global tubular neighborhood. The obstruction lies in $\mathrm{Ext}^1(N_{X/Y}, T_X)$.

Before the proof of Theorem 5.2.4, we introduce the following notation. If $Z \subset Y$ is a 1-dimensional subscheme corresponding to a point in the fibre

$I_n(Y,C)$ of Hilbert–Chow, we can attach to Z its “finite part”, the finite subset $F_Z \subset Z$ which is the support of the maximal zero-dimensional subsheaf of $\mathscr{O}_Z$, namely the quotient $\mathscr{I}_C/\mathscr{I}_Z$.

Proof of Theorem 5.2.4. By [70, Cor. 12.9] the Hilbert–Chow map is a local isomorphism around normal schemes, so we may identify an open neighborhood of the cycle of C in the Chow scheme with an open neighborhood U of $[C]$ in the Hilbert scheme $I_{1-g}(Y,\beta)$. We then consider the Hilbert–Chow map

$$\mathsf{h} = \mathsf{h}_{1-g+n} : I_{1-g+n}(Y,\beta) \to \mathrm{Chow}_1(Y,\beta)$$

and we fix a point in the fibre $[Z_0] \in I_n(Y,C)$. It is easy to reduce to the case where the finite part $F_0 = F_{Z_0} \subset Z_0$ is confined on C, that is, Z_0 has only embedded points. We need to show that the Hilbert scheme is locally analytically isomorphic to $U \times I_n(Y,C)$ about $[Z_0]$. By Lemma 5.3.3, the universal embedding $\mathscr{C} \subset Y \times U$, locally around the finite set of points $F_0 \subset C \subset \mathscr{C}$, is locally analytically isomorphic to the embedding of the zero section $C \times U \subset C \times U \times \mathbb{C}^2$ of the trivial rank 2 bundle. In particular we can find, in $C \times U \times \mathbb{C}^2$ and in $Y \times U$, analytic open neighborhoods V and V' of F_0, fitting in a commutative diagram

$$\begin{tikzcd}
(C \times U) \cap V \arrow[r, hook] \arrow[d, "\wr"] & V \arrow[r, hook, "\text{open}"] \arrow[d, "\wr"] & C \times U \times \mathbb{C}^2 \\
\mathscr{C} \cap V' \arrow[r, hook] & V' \arrow[r, hook, "\text{open}"] & Y \times U
\end{tikzcd}$$

where the vertical maps are analytic isomorphisms. Now consider the open subset

$$A = \left\{ (Z,u) \in I_n(Y,C) \times U \,\middle|\, F_Z \subset V_u \right\} \subset I_n(Y,C) \times U.$$

Letting φ denote the isomorphism $V \xrightarrow{\sim} V'$, given a pair $(Z,u) \in A$ we can look at $Z' = \mathscr{C}_u \cup \varphi(F_Z)$, which is a new subscheme of Y, mapping to u under Hilbert–Chow. The association $(Z,u) \mapsto Z'$ defines an isomorphism between A and the open subset $B \subset \mathsf{h}^{-1}(U)$ parametrizing subschemes $Z' \subset Y$ such that $F_{Z'}$ is contained in V'_u, where u is the image of $[Z']$ under Hilbert–Chow. Note that $[Z_0] \in B$ corresponds to $(Z_0, C) \in A$ under this isomorphism. The theorem is proved. $\square$

5.4 The DT theory of an Abel–Jacobi curve

In this section we fix a non-hyperelliptic curve C of genus 3, embedded in its Jacobian

$$Y = (\operatorname{Jac} C, \Theta)$$

via an Abel–Jacobi map. We let $\beta = [C] \in H_2(Y,\mathbb{Z})$ be the corresponding curve class. For $n \geq 0$, we let

$$\mathcal{H}^n_C \subset I_{n-2}(Y,\beta)$$

be the component of the Hilbert scheme parametrizing subschemes $Z \subset Y$ whose fundamental cycle is algebraically equivalent to $[C]$.

Let $-1 : Y \to Y$ be the automorphism $y \mapsto -y$, and let $-C$ denote the image of C. As C is non-hyperelliptic, the cycle of C is not algebraically equivalent to the cycle of $-C$ [21]. The Hilbert scheme $I_{n-2}(Y,\beta)$ consists of two connected components, which are interchanged by -1. Moreover, the Abel–Jacobi embedding $C \subset Y$ has unobstructed deformations, and there is an isomorphism $Y \xrightarrow{\sim} \mathcal{H}_C^0$ given by translations [45].

Example 5.4.1. As remarked in [33, Example 2.3], the morphism

$$\mathcal{H}_C^1 \to \mathcal{H}_C^0 \times Y$$

sending $T_x(C) \cup y \mapsto (T_x(C), y)$, where T_x denotes translation by x, is the Albanese map. It can be easily checked that $\mathcal{H}_C^1$ is isomorphic to the blow-up

$$\mathrm{Bl}_{\mathcal{U}}(\mathcal{H}_C^0 \times Y),$$

where $\mathcal{U}$ is the universal family. In particular, $\mathcal{H}_C^1$ is smooth of dimension 6. $\diamondsuit$

The quotient of the Hilbert scheme by the translation action of Y gives a Deligne–Mumford stack $I_m(Y,\beta)/Y$. In fact, since the Y-action is free, this is an algebraic space. The *reduced* Donaldson–Thomas invariants

$$\mathsf{DT}_{m,\beta}^Y = \int_{I_m(Y,\beta)/Y} \nu \, \mathrm{d}\chi \in \mathbb{Q}$$

were introduced in [20] for arbitrary abelian threefolds. We consider their generating function

$$\mathsf{DT}_\beta(p) = \sum_{m \in \mathbb{Z}} \mathsf{DT}_{m,\beta}^Y p^m.$$

We state the following result as a corollary of Theorem 5.2.3.

COROLLARY 5.4.2. *Let $C \subset Y$ be non-hyperelliptic, embedded in class β. Then*

$$\mathsf{DT}_\beta(p) = 2p^{-2}(1+p)^4.$$

PROOF. As the Hilbert–Chow morphism is an isomorphism around normal schemes, we have an isomorphism

$$I_{-2}(Y,\beta) \xrightarrow{\sim} \mathrm{Chow}_1(Y,\beta).$$

On the other hand, the Hilbert scheme is the disjoint union of two copies of $\mathcal{H}_C^0$, where $\mathcal{H}_C^0 \cong Y$ because C is not hyperelliptic. Focusing on the component parametrizing translates of C, the Hilbert–Chow morphism $\mathcal{H}_C^n \to \mathcal{H}_C^0$ induces an isomorphism

$$Y \times \mathrm{Quot}_n(\mathscr{I}_C) \xrightarrow{\sim} \mathcal{H}_C^n$$

by Theorem 5.2.3. This shows that the quotient space $\mathcal{H}_C^n/Y$ is isomorphic to the Quot *scheme* $\mathrm{Quot}_n(\mathscr{I}_C)$. Keeping into account the second component of $I_{n-2}(Y,\beta)$, still isomorphic to $\mathcal{H}_C^n$, we find

$$\mathsf{DT}_{n-2,\beta}^Y = 2 \cdot \tilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C)),$$

where $\tilde{\chi}$ denotes the Behrend weighted Euler characteristic. Then

$$\mathsf{DT}_\beta(p) = \sum_{n\geq 0} \mathsf{DT}^Y_{n-2,\beta} p^{n-2} = 2p^{-2}\sum_{n\geq 0}\tilde{\chi}(\mathrm{Quot}_n(\mathscr{I}_C))p^n = 2p^{-2}(1+p)^4,$$

where the last equality follows from Proposition 4.5.6. ☐

If one considers homology classes of type $(1,1,d)$ for all $d\geq 0$, on an arbitrary abelian threefold Y, one has the formula

$$\sum_{d\geq 0}\sum_{m\in\mathbb{Z}} \mathsf{DT}^Y_{m,(1,1,d)}(-p)^m q^d = -K(p,q)^2, \tag{5.4.1}$$

where K is the Jacobi theta function

$$K(p,q) = (p^{1/2}-p^{-1/2})\prod_{m\geq 1}\frac{(1-pq^m)(1-p^{-1}q^m)}{(1-q^m)^2}.$$

Relation (5.4.1) was conjectured in [20] and proved in [59, 61]. Corollary 5.4.2 confirms the coefficient of q via Quot schemes, when Y is the Jacobian of a general curve. Indeed, in this case the Abel–Jacobi class is of type $(1,1,1)$.

The *local* DT theory of a general Abel–Jacobi curve C of genus 3 is determined as follows. Using again the isomorphism $Y \cong \mathcal{H}^0_C$, we can compute the BPS number

$$n_{3,C} = \nu(\mathscr{I}_C) = -1,$$

thus the DT/PT correspondence at C (Theorem 5.1.1) yields

$$\mathsf{DT}_C(q) = \mathsf{PT}_C(q) = -q^{-2}(1+q)^4.$$

In other words, the global theory is related to the local one by

$$\mathsf{DT}_\beta(q) = -2\cdot\mathsf{DT}_C(q).$$

Part III

MOTIVIC DT INVARIANTS

6 A VIRTUAL MOTIVE FOR THE QUOT SCHEME

6.1 Introduction

In this chapter we prove that the Quot scheme

$$Q_L^n = \mathrm{Quot}_n(\mathscr{I}_L)$$

is a scheme-theoretic critical locus, in the sense of Definition 1.2.1. Here L is a line in the local Calabi–Yau threefold $\mathbb{A}^3$. This result is the first of a series of similarities between Q_L^n and $\mathrm{Hilb}^n(\mathbb{A}^3)$, that we will keep exploring in the next chapter. From the critical locus structure we obtain a canonical virtual motive

$$[Q_L^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

via motivic vanishing cycles, as explained in Section 2.1.3. We end the chapter by proving that the above motive lives in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$.

6.2 The Quot scheme as a critical locus

Let $\mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x,y,z]$ be affine space. Let V be a fixed n-dimensional complex vector space. To turn V into a $\mathbb{C}[x,y,z]$-module one needs to specify three *pairwise commuting* endomorphisms of V (up to simultaneous conjugation).

Let now $\mathscr{I}_L = (x,y) \subset \mathbb{C}[x,y,z]$ be the ideal of the line

$$L:\; x = y = 0 \;\text{ in } \mathbb{A}^3,$$

and set $\mathrm{GL}_n = \mathrm{GL}(V)$. Let $(A,B,C) \in \mathrm{End}(V)^3$ define a $\mathbb{C}[x,y,z]$-module structure on V, and let us fix a $\mathbb{C}$-linear map $\phi : \mathscr{I}_L \to V$. Then ϕ determines two vectors $a = \phi(x)$ and $b = \phi(y)$ and we observe that

- ϕ is $\mathbb{C}[x,y,z]$-linear if and only if $A \cdot b = B \cdot a$, and
- ϕ is surjective if and only if the vectors a and b span V as a $\mathbb{C}[x,y,z]$-module.

As multiplication by A, B and C is precisely the $\mathbb{C}[x,y,z]$-linear action of x, y and z on V, and since the polynomial ring is spanned by monomials, the second condition can be rephrased as

$$V = \mathrm{Span}_{\mathbb{C}} \left\{ A^\alpha B^\beta C^\gamma \cdot a, A^\alpha B^\beta C^\gamma \cdot b \,\middle|\, \alpha,\beta,\gamma \geq 0 \right\}.$$

Using the notation of Definition 2.3.2, we could say that ϕ is surjective if and only if $(A,B,C,\phi(x),\phi(y))$ lies in the open set

$$U_n \subset \mathcal{R}_n = \operatorname{End}(V)^3 \times V^2.$$

Notation 6.2.1. We denote by $\mathcal{L}_n$ the closed subscheme

$$\mathcal{L}_n = \left\{ (A,B,C,a,b) \,\middle|\, A\cdot b = B\cdot a \right\} \subset \mathcal{R}_n$$

cut out by the above "linearity condition". We form the locally closed subscheme

$$T_n = \mathcal{L}_n \cap U_n \subset \mathcal{R}_n,$$

and we let

$$\ell_n : T_n \subset \mathcal{R}_n \to \mathbb{A}^1$$

denote the restriction of the trace function $(A,B,C,a,b) \mapsto \operatorname{Tr} A[B,C]$, first introduced in (2.3.2). ♦

Recall (from Lemma 2.3.4) that the GL_n-action on $\mathcal{R}_n$ given by

$$g\cdot(A,B,C,a,b) = (A^g, B^g, C^g, ga, gb)$$

is free on U_n, and the geometric quotient

$$\overline{U}_n = U_n / \mathrm{GL}_n = \mathcal{R}_n /\!\!/_{\det} \mathrm{GL}_n$$

is a smooth quasi-projective variety (which we interpreted as the moduli space of 2-framed n-dimensional representations of the three loop quiver in Section 2.3). Since $T_n \subset U_n$ is a closed invariant subscheme, the quotient map $U_n \to \overline{U}_n$ restricts to a geometric quotient

$$\pi : T_n \to \overline{T}_n = T_n / \mathrm{GL}_n = \mathcal{L}_n /\!\!/_{\det} \mathrm{GL}_n\,.$$

LEMMA 6.2.1. *The schemes T_n and $\overline{T}_n$ are smooth of dimension $3n^2+n$ and $2n^2+n$ respectively.*

PROOF. Let us fix coordinates $(A_{ij}, B_{ij}, C_{ij}, a_k, b_l)$ on $\mathcal{R}_n$. Then $\mathcal{L}_n \subset \mathcal{R}_n$ is cut out by n quadratic polynomials

$$p_i = \sum_{j=1}^n A_{ij} b_j - B_{ij} a_j, \qquad 1 \le i \le n.$$

Let $x = (A,B,C,a,b) \in \mathcal{L}_n$ be a point. The jacobian matrix at x is an $n \times (3n^2+2n)$-matrix of the form $J_x = (N \,|\, -B \,|\, A)$, where the i-th row of

$$N = \begin{pmatrix} b & 0 & \cdots & 0 & -a & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 & 0 & -a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b & 0 & 0 & \cdots & -a \end{pmatrix}$$

is filled in by the derivatives of p_i with respect to A_{kj} and B_{kj}. (The $n^2 \times n^2$ block of zeros corresponding to derivatives with respect to C_{kj} has been omitted, and we view a and b as row vectors.) If $x \in U_n$, the vectors a and b cannot both be zero. Then the Jacobian matrix J_x evaluated at a point $x \in T_n$ must have a nonzero entry in every row; this shows that T_n avoids the singular locus of $\mathcal{L}_n$, in particular it is smooth of dimension $\dim \mathcal{R}_n - n = 3n^2 + n$. Since GL_n acts with trivial stabilizers, T_n / GL_n is smooth as well, and of dimension $2n^2 + n$. □

We observe that Q_L^n is set-theoretically a critical locus before proving the scheme-theoretic statement. As a set, Q_L^n is described as follows:

$$\begin{aligned} Q_L^n &= \left\{ \mathbb{C}[x,y,z]\text{-linear epimorphisms } \mathscr{I}_L \twoheadrightarrow V \right\} \big/ \mathrm{GL}_n \\ &= \left\{ (A,B,C,a,b) \in T_n \,\middle|\, A,B,C \text{ pairwise commute} \right\} \big/ \mathrm{GL}_n . \end{aligned}$$

The function ℓ_n is GL_n-invariant, so it descends to the quotient.

Definition 6.2.2. We let $f_n : \overline{T}_n \to \mathbb{A}^1$ be the regular function extending ℓ_n. ♢

The condition $\mathrm{d} f_n = 0$ says precisely that the three matrices pairwise commute, so closed points of Q_L^n correspond to closed points of $Z(\mathrm{d} f_n) \subset \overline{T}_n$. We will show that $Q_L^n = Z(\mathrm{d} f_n)$ as a scheme in Theorem 6.2.5 below. Before doing so, we give an alternative description of the spaces T_n and $\overline{T}_n$.

6.2.1 Non-commutative Hilbert and Quot schemes

In (2.3.1) we introduced the *non-commutative Hilbert scheme* via geometric invariant theory. We briefly recall why it deserves this name, and then we give an analogue on the Quot scheme side. In this whole section,

$$R = \mathbb{C}\langle x, y, z \rangle$$

is the free (non-commutative) $\mathbb{C}$-algebra on three generators, and for a complex scheme B, we denote by R_B the sheaf of $\mathscr{O}_B$-algebras associated to the presheaf

$$R \otimes_{\mathbb{C}} \mathscr{O}_B = \mathscr{O}_B \langle x, y, z \rangle .$$

Non-commutative Hilbert scheme

One can construct a functor $\mathcal{H}_R^n : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Sets}$ by sending a complex scheme B to the set of equivalence classes of triples (M, v, β), where M is a left R_B-module which is locally free of rank n as an $\mathscr{O}_B$-module, $v \in \Gamma(B, M)$ generates M as an R_B-module and $\beta \subset \Gamma(B, M)$ is a basis of M as an $\mathscr{O}_B$-module. The equivalence relation is defined in the obvious way: one has $(M, v, \beta) \sim (M', v', \beta')$ when there is an $\mathscr{O}_B$-linear isomorphism $\Phi : M \xrightarrow{\sim} M'$ taking β to β' and v to v'. The functor just described is represented by the quasi-affine smooth complex scheme that we denoted U_n^1 (Definition 2.3.2, p. 20). Note that the pair (M, v) determines and is determined by an R_B-linear surjection $\theta : R_B \twoheadrightarrow M$, with $v = \theta(1)$.

One can also consider the functor $\overline{\mathcal{H}}_R^n$ sending a scheme B to the set of equivalence classes of pairs (M, v), where M and v are just as above, but no choice of basis is made. Again, we declare that $(M, v) \sim (M', v')$ when there is an $\mathscr{O}_B$-linear isomorphism $\Phi : M \xrightarrow{\sim} M'$ taking v to v'.

THEOREM 6.2.3. *The scheme U_n^1 represents the functor $\mathcal{H}_R^n$. There is a scheme* Hilb_R^n *representing $\overline{\mathcal{H}}_R^n$, and the forgetful morphism $U_n^1 \to \mathrm{Hilb}_R^n$ is a universal categorical quotient and a principal* GL_n*-bundle. In particular, one has an isomorphism of schemes*

$$U_n^1 / \mathrm{GL}_n \xrightarrow{\sim} \mathrm{Hilb}_R^n.$$

We refer to [46, Theorem 2.7] for a proof of this result in a more general setting (more precisely, for finitely generated associative algebras $\mathcal{A}$ over a commutative ring k). See also [58] for a proof in the case where R gets replaced by $\mathbb{Z}\langle x_1, \ldots, x_m\rangle$ and [80, 29] for a version of the result where the functors are represented by algebras (and not schemes). Note that Hilb_R^n can be seen as the moduli space of left ideals $J \subset R$ of codimension n (that is, such that R/J has dimension n as a $\mathbb{C}$-vector space). Indeed, the equivalence relation $\sim$ identifies two quotients $R_B \twoheadrightarrow M$ and $R_B \twoheadrightarrow M'$ precisely when they have the same kernel. Therefore, the scheme

$$\mathrm{Hilb}_R^n \cong U_n^1 / \mathrm{GL}_n$$

deserves to be called non-commutative Hilbert scheme.

Non-commutative Quot scheme

We now let the ideal $K = \langle x, y\rangle \subset R$ take the role played by the $\mathbb{C}$-algebra R in the previous paragraph. This gives rise to a notion of "non-commutative Quot scheme", as we now explain. For a complex scheme B, let K_B denote the submodule

$$K_B = K \otimes_{\mathbb{C}} \mathscr{O}_B \subset R_B.$$

Consider the functor $\mathcal{Q}_K^n : \mathsf{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathsf{Sets}$ defined by

$$B \mapsto \left\{ \langle M, \theta, \beta\rangle \;\middle|\; \begin{array}{c} M \text{ is a left } R_B\text{-module, locally free of rank } n \text{ over } \mathscr{O}_B, \\ \theta : K_B \twoheadrightarrow M \text{ is an } R_B\text{-linear epimorphism} \\ \text{and } \beta \subset \Gamma(B, M) \text{ is a basis of } M \text{ as an } \mathscr{O}_B\text{-module} \end{array} \right\}.$$

Here $\langle M, \theta, \beta\rangle$ denotes the equivalence class of the triple (M, θ, β), where we declare $(M, \theta, \beta) \sim (M', \theta', \beta')$ when one has a commutative diagram

$$\begin{array}{ccc} K_B & \xrightarrow{\theta} & M \\ \| & & \downarrow{\scriptstyle \Phi} \\ K_B & \xrightarrow{\theta'} & M' \end{array} \tag{6.2.1}$$

with Φ an $\mathscr{O}_B$-linear isomorphism transforming β into β'. One can also define the functor $\overline{\mathcal{Q}}_K^n : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Sets}$ just as above but forgetting the choice of a basis, namely by letting

$$B \mapsto \left\{ \langle M, \theta \rangle \;\middle|\; \begin{array}{c} M \text{ is a left } R_B\text{-module, locally free of rank } n \text{ over } \mathscr{O}_B, \\ \text{and } \theta : K_B \twoheadrightarrow M \text{ is an } R_B\text{-linear epimorphism} \end{array} \right\}.$$

Here we declare that $(M, \theta) \sim (M', \theta')$ when there is a commutative diagram as in (6.2.1). Notice that, by considering the kernel of the surjection, a pair $\langle M, \theta \rangle$ uniquely determines a left ideal $\mathcal{I} \subset K_B$ (such that the quotient $K_B/\mathcal{I}$ is a locally free $\mathscr{O}_B$-module).

The next result is the "Quot" analogue of Theorem 6.2.3. The proof follows [46, Section 2] closely.

THEOREM 6.2.4. *The scheme T_n represents the functor $\mathcal{Q}_K^n$, and the quotient $\overline{T}_n$ represents $\overline{\mathcal{Q}}_K^n$.*

PROOF. Let $V = \mathbb{C}^n$ with its standard basis $e_1, \ldots, e_n$. Consider the free module $M_0 = V \otimes_{\mathbb{C}} \mathscr{O}_{\mathcal{R}_n}$ with basis $\beta_0 = \{ e_j \otimes 1 : 1 \le j \le n \}$. Let $(X_{ij}, Y_{ij}, Z_{ij}, u_k, w_l)$ be the coordinates on the affine space $\mathcal{R}_n$. Then M_0 has distinguished elements $v_x = \sum e_j \otimes u_k$ and $v_y = \sum e_j \otimes w_l$. Let $\theta_0 : K_{\mathcal{R}_n} \to M_0$ be the map given by $\theta_0(x) = v_x$ and $\theta_0(y) = v_y$. Restricting the triple (M_0, θ_0, β_0) to $T_n \subset \mathcal{R}_n$ gives a morphism of functors

$$T_n \to \mathcal{Q}_K^n,$$

whose inverse is constructed as follows. Let B be a scheme, set again $V = \mathbb{C}^n$ and fix a B-valued point $\langle M, \theta, \beta \rangle \in \mathcal{Q}_K^n(B)$. The R-action on $\beta \subset \Gamma(B, M)$ determines three endomorphisms $(X, Y, Z) : B \to \mathrm{End}(V)^3$ and the images of x and y under the map $\theta : K_B \twoheadrightarrow M$ correspond to a morphism $(u, w) : B \to V^2$. The R_B-linearity of θ says that $(X, Y, Z, u, w) : B \to \mathcal{R}_n$ factors through the subscheme $\mathcal{L}_n \subset \mathcal{R}_n$ cut out by $X \cdot w = Y \cdot u$, and the surjectivity of θ says that it actually factors through $\mathcal{L}_n \cap U_n = T_n$. Therefore T_n represents $\mathcal{Q}_K^n$.

Next, let $\pi : T_n \to \overline{T}_n$ be the quotient map, which we know is a principal GL_n-bundle. This implies that $\pi^* : \mathsf{QCoh}(\overline{T}_n) \xrightarrow{\sim} \mathsf{QCoh}_{\mathrm{GL}_n}(T_n)$ is an equivalence of categories, preserving locally free sheaves [46, Prop. 4.5]. Consider the universal triple $\langle M_0, \theta_0, \beta_0 \rangle$ defined above. Then M_0 is a GL_n-equivariant vector bundle on T_n; it follows that, up to isomorphism, there is a unique locally free sheaf $\mathscr{M}$ on $\overline{T}_n$ such that $\pi^* \mathscr{M} \cong M_0$. In fact, $\mathscr{M} \cong (\pi_* M_0)^{\mathrm{GL}_n} \subset \pi_* M_0$, the subsheaf of GL_n-invariant sections. The two sections v_x and v_y, being GL_n-invariant, descend to sections of $\mathscr{M}$, still denoted v_x, v_y. These generate $\mathscr{M}$ as an $R_{\overline{T}_n}$-module, so we get a surjection $\vartheta : K_{\overline{T}_n} \twoheadrightarrow \mathscr{M}$ sending $x \mapsto v_x$ and $y \mapsto v_y$. In particular, the pair $\langle \mathscr{M}, \vartheta \rangle$ defines a morphism of functors

$$\overline{T}_n \to \overline{\mathcal{Q}}_K^n.$$

We now construct its inverse. Let B be a scheme and fix a B-valued point $\langle N, \theta \rangle \in \overline{\mathcal{Q}}_K^n(B)$. Let $(B_i : i \in I)$ be an open cover of B such that $N_i = N|_{B_i}$ is free of rank n over $\mathscr{O}_{B_i}$. Choose a basis $\beta_i \subset \Gamma(B_i, N_i)$ and let $v_{x,i} = \theta(x)|_{B_i}$

be the restriction of $\theta(x) \in \Gamma(B,N)$ to N_i. Define $v_{y,i}$ similarly for all $i \in I$. As usual, the pair $(v_{x,i}, v_{y,i})$ defines a linear surjection $\theta_i : K_{B_i} \twoheadrightarrow N_i$. Each triple $\langle N_i, \theta_i, \beta_i \rangle$ then defines a point $\psi_i : B_i \to T_n$, and for all indices i and j there is a matrix $g \in \mathrm{GL}_n(\mathscr{O}_{B_{ij}})$ sending β_i to β_j. In other words, g defines a map $g : B_{ij} \to \mathrm{GL}_n$ such that $g \cdot \psi_i = \psi_j$. Then $\pi \circ \psi_i$ and $\pi \circ \psi_j$ agree on B_{ij}, and this determines a unique map $p : B \to \overline{T}_n$ such that $(N,\theta) \sim p^*(\mathscr{M}, \vartheta)$. This shows that $\overline{T}_n$ represents $\overline{\mathcal{Q}}_K^n$. □

The upshot is that the B-valued points of $\overline{T}_n$ can now be identified with left ideals $\mathcal{I} \subset R_B$ contained in K_B (such that $K_B/\mathcal{I}$ is a locally free $\mathscr{O}_B$-module of rank n).

Notation 6.2.2. By analogy with “Hilb”, where we sometimes write Hilb_R^n for the quotient U_n^1/GL_n (justified by Theorem 6.2.3), on the “Quot” side we may write

$$\mathrm{Quot}_K^n$$

for the scheme that we previously denoted $\overline{T}_n = T_n/\mathrm{GL}_n = \mathcal{L}_n /\!\!/_{\det} \mathrm{GL}_n$. By Theorem 6.2.4, Quot_K^n could be called a *non-commutative Quot scheme.* ◆

Recall the trace potential f_n of Definition 6.2.2, defined on $\overline{T}_n$.

THEOREM 6.2.5. *There is a closed immersion*

$$\mathrm{Quot}_n(\mathscr{I}_L) \hookrightarrow \overline{T}_n = \mathrm{Quot}_K^n$$

cut out scheme-theoretically by the exact one-form $\mathrm{d} f_n$.

PROOF. Let B be a scheme. Observe that there is an inclusion of sets

$$\mathrm{Quot}_n(\mathscr{I}_L)(B) \subset \mathrm{Quot}_K^n(B).$$

A B-valued point $[\mathcal{I}]$ of the non-commutative Quot scheme defines a B-valued point of the commutative Quot scheme if and only if the R-action on the corresponding ideal $\mathcal{I}$ descends to a $\mathbb{C}[x,y,z]$-action. This happens precisely when the actions of x, y and z on $\mathcal{I}$ commute with each other. Let then $W \subset \mathrm{Quot}_K^n$ be the image of the zero locus

$$\big\{ (X,Y,Z,v,w) \,\big|\, [X,Y] = [X,Z] = [Y,Z] = 0 \big\} \subset T_n$$

under the quotient map. Then $[\mathcal{I}]$ belongs to $\mathrm{Quot}_n(\mathscr{I}_L)(B)$ if and only if the corresponding morphism $B \to \mathrm{Quot}_K^n$ factors through W. But W agrees, as a scheme, with the critical locus of f_n, by [71, Prop. 3.8]. □

COROLLARY 6.2.6. *The function f_n induces a canonical relative virtual motive*

$$[Q_L^n]_{\mathrm{relvir}} = -\mathbb{L}^{-(2n^2+n)/2}[\phi_{f_n}]_{Q_L^n} \in \mathcal{M}_{Q_L^n}^{\hat{\mu}}$$

on the Quot scheme $Q_L^n = \mathrm{Quot}_n(\mathscr{I}_L)$.

PROOF. By Lemma 6.2.1, $\overline{T}_n$ is smooth of dimension $2n^2+n$. Then the general construction recalled in Section 2.1.3 applies. □

We will denote by

$$[Q_L^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

the associated absolute virtual motive. We will soon study these classes more closely.

Example 6.2.7. The non-commutative Hilbert scheme $\mathrm{Hilb}_R^n = U_n^1 / \mathrm{GL}_n$ introduced in (2.3.1) has dimension $2n^2 + n = \dim \overline{T}_n$. The trace functions cutting out the Quot scheme and the Hilbert scheme are exactly the same, hence $\mathrm{Quot}_n(\mathscr{I}_L)$ and $\mathrm{Hilb}^n(\mathbb{A}^3)$ have the same expected dimension. The Hilbert scheme is nonsingular if $n \le 3$ and singular otherwise, whereas Q_L^n is already singular if $n \ge 2$, see Example 4.3.12. Let us fix $n = 1$. In this case the trace functions vanish so the virtual motives are a shift of the naive motives by $\mathbb{L}^{-3/2}$. On the Hilbert scheme side we have

$$[\mathrm{Hilb}^1(\mathbb{A}^3)]_{\mathrm{vir}} = \mathbb{L}^{-3/2} \cdot \mathbb{L}^3 = \mathbb{L}^{3/2},$$

while on the Quot side we have

$$\begin{aligned} [Q_L^1]_{\mathrm{vir}} &= \mathbb{L}^{-3/2} \cdot [\mathrm{Bl}_L \mathbb{A}^3] \\ &= \mathbb{L}^{-3/2} \cdot \left([\mathbb{A}^3 \setminus L] + [L \times \mathbb{P}^1]\right) \\ &= \mathbb{L}^{3/2} + \mathbb{L}^{1/2}. \end{aligned} \qquad \diamondsuit$$

We end this chapter by showing that $[Q_L^n]_{\mathrm{vir}}$ is a "monodromy-free" class, thanks to the good equivariance properties of the trace function.

6.2.2 Equivariance of the family

Consider, for $m \ge 0$, the subset $S_m \subset \mathscr{O}_{\mathcal{R}_n}(\mathcal{R}_n)$ of functions h satisfying $h(g \cdot P) = (\det g)^m h(P)$ for $g \in \mathrm{GL}_n$ and $P \in \mathcal{R}_n$. Then we have

$$\overline{U}_n = \mathrm{Proj} \bigoplus_{m \ge 0} S_m.$$

By general GIT, the natural inclusion $\mathscr{O}_{\mathcal{R}_n}(\mathcal{R}_n)^{\mathrm{GL}_n} \subset \bigoplus_{m\ge 0} S_m$ induces a *projective* morphism

$$\mathsf{p}_n : \overline{U}_n \to Y_0 \tag{6.2.2}$$

where the affine scheme $Y_0 = \mathrm{Spec}\, \mathscr{O}_{\mathcal{R}_n}(\mathcal{R}_n)^{\mathrm{GL}_n} = \mathcal{R}_n /\!\!/_0 \mathrm{GL}_n$ can be viewed as the GIT quotient at the trivial character. The following result is an application of Theorem 2.1.16.

THEOREM 6.2.8. *One has the relation*

$$[\phi_{f_n}] = [f_n^{-1}(1)] - [f_n^{-1}(0)] \in \mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}.$$

In particular, $[Q_L^n]_{\mathrm{vir}}$ lies in $\mathcal{M}_{\mathbb{C}}$. Moreover, if $\mathsf{a} : Q_L^n \to \widetilde{Q}_L^n$ is the affinization map, we have

$$\mathsf{a}_! [\phi_{f_n}]_{Q_L^n} \in \mathcal{M}_{\widetilde{Q}_L^n} \subset \mathcal{M}_{\widetilde{Q}_L^n}^{\hat{\mu}}.$$

PROOF. The three-dimensional torus $\mathbf{T} = \mathbb{G}_m^3$ acts on T_n by

$$t \cdot (A, B, C, a, b) = (t_1 A, t_2 B, t_3 C, t_1 t_3 a, t_2 t_3 b).$$

Since this action commutes with the GL_n-action, it descends to the quotient $\overline{T}_n$. Moreover, the trace function $\ell_n : T_n \to \mathbb{A}^1$ is $\mathbf{T}$-equivariant with respect to the primitive character $\chi(t) = t_1 t_2 t_3$. In other words, for all $P \in T_n$, we have $\ell_n(t \cdot P) = \chi(t)\ell_n(P)$, and similarly for f_n. The induced action on $\overline{T}_n$ by the diagonal torus $\mathbb{G}_m \subset \mathbf{T}$ is circle compact, that is, it has compact fixed locus and the limits $\lim_{t\to 0} t \cdot P$ exist in $\overline{T}_n$ for all $P \in \overline{T}_n$. To see this, notice that the restriction of (6.2.2) to the closed subscheme $\overline{T}_n$ results in a projective $\mathbb{G}_m$-equivariant map $\mathsf{p}_n : \overline{T}_n \to Y_0$, and the proof of [7, Lemma 3.4] shows that Y_0 has a unique $\mathbb{G}_m$-fixed point, and all orbits have this point in their closure. In other words, limits exist in Y_0. Therefore, by properness of p_n, we conclude that the $\mathbb{G}_m$-fixed locus in $\overline{T}_n$ is compact and limits exist. Then the first statement follows by part (i) of Theorem 2.1.16. In particular, the absolute virtual motive carries no monodromy,

$$[Q_L^n]_{\mathrm{vir}} = -\mathbb{L}^{-(2n^2+n)/2}[\phi_{f_n}] \in \mathcal{M}_{\mathbb{C}}.$$

Finally, the hypersurface $f_n^{-1}(0) = \{\operatorname{Tr} A[B, C] = 0\} \subset \overline{T}_n$ is reduced, as the polynomial

$$\sum_{i,k} A_{ik} \sum_l (B_{kl} C_{li} - C_{kl} B_{li})$$

has no linear factor. The last statement then follows from part (ii) of Theorem 2.1.16. $\square$

7 On the motivic partition function of the Quot scheme

7.1 Introduction

In this chapter we compute the *motivic partition function* of the Quot scheme

$$\mathsf{Z}(t) = \sum_{n\geq 0} [Q_L^n]_{\mathrm{vir}} t^n \in \mathcal{M}_{\mathbb{C}}[\![t]\!] \tag{7.1.1}$$

with two methods. The first one (in Section 7.2) is a direct motivic vanishing cycle calculation, whereas the second one (in Section 7.3) is by a stratification technique which allows us to restrict attention to the (virtual) motives of the deepest strata inside Q_L^n. The latter strategy can be viewed as the motivic analogue of the one we used in Section 4.4 to prove the formula $\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n)$.

Unfortunately, we have not succeeded in writing $\mathsf{Z}(t)$ as an intrinsic function depending only on the Lefschetz motive $\mathbb{L}$. However, we can still use our stratification to *define* a virtual motive

$$[Q_C^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$$

for the Quot scheme Q_C^n of an arbitrary smooth curve C in a smooth quasi-projective threefold Y. Via the power structure on the ring of motivic weights, the corresponding motivic partition function is determined, just like Z, by the virtual motives of the deepest strata in Q_L^n.

A special case is the following. When Y is a *projective* Calabi–Yau threefold and $C \subset Y$ is a smooth curve with BPS number $n_{g,C} = 1$ (for instance, rigid), the above class is a refinement of the numerical DT invariant

$$\mathsf{DT}_{n,C} \in \mathbb{Z},$$

hence can be seen as a *motivic DT invariant* for Y at C.

7.2 Vanishing cycle calculation

We start by stating the main result of this section. Consider the scheme

$$E_n = \left\{ (A,B,a,b) \in C_n \times V^2 \,\middle|\, A\cdot b = B\cdot a \right\} \subset C_n \times V^2,$$

where $C_n \subset \mathrm{End}(V)^2$ is the commuting variety, and define the generating series

$$\mathsf{E}(t) = \sum_{n\geq 0} \frac{[E_n]}{\mathsf{GL}_n} t^n \in K_0(\mathsf{St}_{\mathbb{C}})[\![t]\!].$$

The generating function $\mathsf{C}(t)$ for the motives of the stacks C_n/GL_n is determined by the Feit–Fine formula (Theorem 2.1.4, p. 9). We have the following.

THEOREM 7.2.1. *The motivic partition function of the Quot scheme* (7.1.1) *is given by the formula*

$$\mathsf{Z}(t) = \frac{\mathsf{E}(t\mathbb{L}^{-1/2})}{\mathsf{C}(t\mathbb{L}^{-1/2})}.$$

The proof uses the techniques anticipated in Section 2.3.

Key characters

We summarize in the diagram

$$\begin{tikzcd} & T_n \arrow[r, hook, "\ell_n"] \arrow[d] & \mathcal{R}_n \arrow[d, "\widetilde{\mathsf{W}}_n"] \\ Q_L^n \arrow[r, hook] & \overline{T}_n \arrow[r, "f_n"] & \mathbb{A}^1 \end{tikzcd}$$

some of the notation used so far. Here $Q_L^n = \{\mathrm{d} f_n = 0\}$. We let

$$\widetilde{Y}_n = \widetilde{\mathsf{W}}_n^{-1}(0), \qquad \widetilde{Z}_n = \widetilde{\mathsf{W}}_n^{-1}(1).$$

We already dealt with these objects in Section 2.3. This time we also need to consider

$$Y_n = \widetilde{Y}_n \cap \mathcal{L}_n, \qquad Z_n = \widetilde{Z}_n \cap \mathcal{L}_n,$$

the special and generic fibres of the restricted potential $\mathcal{L}_n \to \mathbb{A}^1$. For $0 \le k \le n$, let

$$X_n^k = \{\, x \in \mathcal{R}_n \mid \mathrm{Span}(x) \text{ is } k\text{-dimensional} \,\} \subset \mathcal{R}_n.$$

We introduced the span of a point x in Definition 2.3.1. Consider

$$Y_n^k = Y_n \cap X_n^k, \qquad Z_n^k = Z_n \cap X_n^k$$

and the motivic differences

$$\omega_n^k = [Y_n^k] - [Z_n^k], \qquad \omega_n = [Y_n] - [Z_n] = \sum_{k=0}^{n} \omega_n^k.$$

We can now start the calculation. Applying Theorem 2.1.16 to the $\mathbb{G}_m$-action on T_n described during the proof of Theorem 6.2.8, we find that $-\omega_n^n = [\phi_{\ell_n}]$, so in particular we can write

$$[Q_L^n]_{\mathrm{vir}} = -\mathbb{L}^{-(2n^2+n)/2} \frac{[\phi_{\ell_n}]}{\mathsf{GL}_n} = \mathbb{L}^{-(2n^2+n)/2} \frac{\omega_n^n}{\mathsf{GL}_n} = \frac{\omega_n^n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!}. \tag{7.2.1}$$

LEMMA 7.2.2. *For $0 \le k \le n$, one has the formula*

$$\omega_n^k = [\mathrm{Gr}(k,V)]\mathbb{L}^{(n-k)(n+2k)}[C_{n-k}]\omega_k^k. \tag{7.2.2}$$

PROOF. First, let us compute the motive of Y_n^k. We need the motive of the fibre of the map $h : Y_n^k \to \mathrm{Gr}(k,V)$ sending a point to its span. We use exactly the same strategy and notation as in Section 2.3. Fix $\Lambda \in \mathrm{Gr}(k,V)$ and choose a basis of V such that the first k vectors of the basis belong to Λ. Then, any $(A,B,C,v,w) \in h^{-1}(\Lambda)$ will be in the form

$$A = \begin{pmatrix} A_0 & A' \\ 0 & A_1 \end{pmatrix},\ B = \begin{pmatrix} B_0 & B' \\ 0 & B_1 \end{pmatrix},\ C = \begin{pmatrix} C_0 & C' \\ 0 & C_1 \end{pmatrix},\ v = \begin{pmatrix} v_0 \\ 0 \end{pmatrix},\ w = \begin{pmatrix} w_0 \\ 0 \end{pmatrix},$$

where A_0, B_0, C_0 are $k \times k$ matrices, A_1, B_1, C_1 are $(n-k)\times(n-k)$ matrices, A', B', C' are $k \times (n-k)$ matrices, and v_0, w_0 are k-vectors. We then find an isomorphism

$$h^{-1}(\Lambda) \cong \mathbb{A}^{3k(n-k)} \times (S \amalg T),$$

where, setting $\mathrm{Tr}_i = \mathrm{Tr}\, A_i[B_i, C_i]$, we let

$$\begin{aligned} S &= \left\{ (A_0,B_0,C_0,v_0,w_0,A_1,B_1,C_1) \,\middle|\, \mathrm{Tr}_0 = \mathrm{Tr}_1 = 0, A_0 \cdot w_0 = B_0 \cdot v_0 \right\}, \\ T &= \left\{ (A_0,B_0,C_0,v_0,w_0,A_1,B_1,C_1) \,\middle|\, \mathrm{Tr}_0 = -\mathrm{Tr}_1 \neq 0, A_0 \cdot w_0 = B_0 \cdot v_0 \right\}. \end{aligned}$$

We also have isomorphisms

$$\begin{aligned} S \times \mathbb{A}^{2(n-k)} &\xrightarrow{\sim} Y_k^k \times \widetilde{Y}_{n-k}, \\ T \times \mathbb{A}^{2(n-k)} &\xrightarrow{\sim} \mathbb{C}^\times \times Z_k^k \times \widetilde{Z}_{n-k}. \end{aligned}$$

The first one is defined by

$$(A_0,B_0,C_0,v_0,w_0,A_1,B_1,C_1;e_1,e_2) \mapsto (A_0,B_0,C_0,v_0,w_0;A_1,B_1,C_1,e_1,e_2),$$

where e_i are $(n-k)$-vectors. The second one is given by

$$\begin{aligned} &(A_0,B_0,C_0,v_0,w_0,A_1,B_1,C_1;e_1,e_2) \\ &\qquad \mapsto (\mathrm{Tr}_0;\mathrm{Tr}_0^{-1} A_0,B_0,C_0,v_0,w_0;\mathrm{Tr}_1^{-1} A_1,B_1,C_1,e_1,e_2). \end{aligned}$$

Therefore, we have

$$\begin{aligned} [Y_n^k] &= [\mathrm{Gr}(k,V)]\mathbb{L}^{3k(n-k)}([S]+[T]) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{3k(n-k)}\mathbb{L}^{-2(n-k)}([Y_k^k][\widetilde{Y}_{n-k}] + (\mathbb{L}-1)[Z_k^k][\widetilde{Z}_{n-k}]) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}([Y_k^k][\widetilde{Y}_{n-k}] + (\mathbb{L}-1)[Z_k^k][\widetilde{Z}_{n-k}]). \end{aligned}$$

To compute the motive of Z_n^k consider the map $l : Z_n^k \to \mathrm{Gr}(k,V)$, defined again by sending a point to its span. The fibre is

$$l^{-1}(\Lambda) \cong \mathbb{A}^{3k(n-k)} \times (S_1 \amalg S_2 \amalg S_3),$$

where S_1, S_2 and S_3 correspond, respectively, to the loci $\mathrm{Tr}_0 = 0$, $\mathrm{Tr}_1 = 0$ and $\mathrm{Tr}_0 = 1 - \mathrm{Tr}_1 \in \mathbb{C}^\times \setminus \{1\}$ inside

$$\left\{ (A_0,B_0,C_0,v_0,w_0,A_1,B_1,C_1) \,\middle|\, \begin{array}{l} \mathrm{Tr}_0 + \mathrm{Tr}_1 = 1,\, A_0 \cdot w_0 = B_0 \cdot v_0, \\ \mathrm{Span}(A_0,B_0,C_0,v_0,w_0) = \mathbb{C}^k \end{array} \right\}.$$

This time we find isomorphisms

$$S_1 \times \mathbb{A}^{2(n-k)} \xrightarrow{\sim} Y_k^k \times \widetilde{Z}_{n-k}$$
$$S_2 \times \mathbb{A}^{2(n-k)} \xrightarrow{\sim} Z_k^k \times \widetilde{Y}_{n-k}$$
$$S_3 \times \mathbb{A}^{2(n-k)} \xrightarrow{\sim} (\mathbb{C}^\times \setminus \{1\}) \times Z_k^k \times \widetilde{Z}_{n-k}$$

allowing us to write

$$\begin{aligned}[Z_n^k] &= [\mathrm{Gr}(k,V)]\mathbb{L}^{3k(n-k)}([S_1]+[S_2]+[S_3]) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}([Y_k^k][\widetilde{Z}_{n-k}]+[Z_k^k][\widetilde{Y}_{n-k}]+(\mathbb{L}-2)[Z_k^k][\widetilde{Z}_{n-k}]).\end{aligned}$$

We can now compute

$$\begin{aligned}\omega_n^k &= [Y_n^k]-[Z_n^k] \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}([Y_k^k][\widetilde{Y}_{n-k}]+(\mathbb{L}-1)[Z_k^k][\widetilde{Z}_{n-k}] \\ &\qquad -[Y_k^k][\widetilde{Z}_{n-k}]-[Z_k^k][\widetilde{Y}_{n-k}]-(\mathbb{L}-2)[Z_k^k][\widetilde{Z}_{n-k}]) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}([Y_k^k]\widetilde{\omega}_{n-k}-[Z_k^k]\widetilde{\omega}_{n-k}) \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}\omega_k^k\widetilde{\omega}_{n-k} \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(3k-2)(n-k)}\omega_k^k[C_{n-k}]\mathbb{L}^{(n-k)(n-k+2)} && \text{by (2.3.7)} \\ &= [\mathrm{Gr}(k,V)]\mathbb{L}^{(n-k)(n+2k)}[C_{n-k}]\omega_k^k.\end{aligned}$$

The formula is proved. □

Proof of Theorem 7.2.1. Recall that $\omega_n = [Y_n]-[Z_n] = \sum_k \omega_n^n$. Then by (7.2.2), and substituting the motive of the Grassmannian (2.1.1), we can write

$$\begin{aligned}\omega_n^n &= \omega_n - \sum_{k=0}^{n-1}[\mathrm{Gr}(k,V)]\mathbb{L}^{(n-k)(n+2k)}[C_{n-k}]\omega_k^k \\ &= \omega_n - [n]_{\mathbb{L}}!\sum_{k=0}^{n-1}\mathbb{L}^{(n-k)(n+2k)}\frac{[C_{n-k}]}{[n-k]_{\mathbb{L}}!}\frac{\omega_k^k}{[k]_{\mathbb{L}}!} \\ &= \omega_n - [n]_{\mathbb{L}}!\sum_{k=0}^{n-1}\widetilde{c}_{n-k}\mathbb{L}^{(n-k)(3n+3k-1)/2}\frac{\omega_k^k}{[k]_{\mathbb{L}}!},\end{aligned}$$

where $\widetilde{c}_i = [C_i]/[\mathrm{GL}_i]$. Thus, dividing out by $\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!$ and using (7.2.1), we find

$$[Q_L^n]_{\mathrm{vir}} = \frac{\omega_n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!} - \sum_{k=0}^{n-1}\widetilde{c}_{n-k}\mathbb{L}^{-(n-k)/2}[Q_L^k]_{\mathrm{vir}}.$$

Rearranging terms,

$$\frac{\omega_n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!} = \sum_{k=0}^{n}\widetilde{c}_{n-k}\mathbb{L}^{-(n-k)/2}[Q_L^k]_{\mathrm{vir}}.$$

Multiplying by t^n and summing over $n \geq 0$ yields

$$\sum_{n\geq 0}\frac{\omega_n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!}t^n = \left(\sum_{n\geq 0}\widetilde{c}_n(t\mathbb{L}^{-1/2})^n\right)\cdot \mathsf{Z}(t),$$

which we may rewrite as

$$Z(t) = \frac{\Omega(t)}{C(t\mathbb{L}^{-1/2})}, \tag{7.2.3}$$

where

$$\Omega(t) = \sum_{n\geq 0} \frac{\omega_n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!} t^n.$$

We now need to compute ω_n. As in the proof of Theorem 6.2.8, the trace map $\mathcal{L}_n \to \mathbb{A}^1$ is $\mathbb{G}_m^3$-equivariant with respect to the primitive character $\chi(t) = t_1 t_2 t_3$ via

$$t \cdot (A, B, C, a, b) = (t_1 A, t_2 B, t_3 C, t_1 t_3 a, t_2 t_3 b),$$

thus according to (2.1.6) one has an isomorphism $Z_n \times \mathbb{G}_m \cong \mathcal{L}_n \setminus Y_n$, inducing the motivic relation

$$[\mathcal{L}_n] = (\mathbb{L}-1)[Z_n] + [Y_n].$$

On the other hand, $[\mathcal{L}_n] = \mathbb{L}^{n^2}[B_n]$, where

$$B_n = \left\{ (A, B, a, b) \,\middle|\, A\cdot b = B \cdot a \right\} \subset \mathrm{End}(V)^2 \times V^2.$$

Define the subscheme

$$E_n = \left\{ (A, B, a, b) \,\middle|\, [A, B] = 0, A \cdot b = B \cdot a \right\} \subset B_n.$$

We can split Y_n as $Y_n' \amalg Y_n''$, where Y_n' is defined by the condition $[A, B] = 0$ and Y_n'' is its complement. Then the map $Y_n \to B_n$ forgetting C splits as a Zariski fibration $Y_n' \to E_n$ with fibre $\mathbb{A}^{n^2}$, and a hyperplane bundle $Y_n'' \to B_n \setminus E_n$, with fibre $\mathbb{A}^{n^2-1}$. So we can write

$$[Y_n] = \mathbb{L}^{n^2}[E_n] + \mathbb{L}^{n^2-1}([B_n] - [E_n]).$$

Using that $\mathbb{L}^{n^2}[B_n] = (\mathbb{L}-1)[Z_n] + [Y_n]$, we find

$$\begin{aligned}
(1-\mathbb{L})\omega_n &= \mathbb{L}^{n^2}[B_n] - \mathbb{L}[Y_n] \\
&= \mathbb{L}^{n^2}[B_n] - \mathbb{L}(\mathbb{L}^{n^2}[E_n] + \mathbb{L}^{n^2-1}[B_n] - \mathbb{L}^{n^2-1}[E_n]) \\
&= (1-\mathbb{L})\mathbb{L}^{n^2}[E_n].
\end{aligned}$$

We conclude that

$$\omega_n = \mathbb{L}^{n^2}[E_n]. \tag{7.2.4}$$

Define the series

$$\mathsf{E}(t) = \sum_{n\geq 0} \frac{[E_n]}{\mathrm{GL}_n} t^n.$$

By formula (7.2.4) we have

$$\frac{\omega_n}{\mathbb{L}^{3n^2/2}[n]_{\mathbb{L}}!} = \mathbb{L}^{-n^2/2} \frac{[E_n]}{[n]_{\mathbb{L}}!} = \mathbb{L}^{-n/2} \frac{[E_n]}{\mathrm{GL}_n}.$$

Hence the remaining factor we needed is the series

$$\Omega(t) = \sum_{n\geq 0} \mathbb{L}^{-n/2} \frac{[E_n]}{\mathrm{GL}_n} t^n = \mathsf{E}(t\mathbb{L}^{-1/2}).$$

By (7.2.3), the proof of the theorem is complete. $\square$

Ideally, we would like to express $\mathsf{E}(t)$ as an "intrinsic" infinite product, involving only (rational functions of) the Lefschetz motive. Before attempting the computation of $\mathsf{E}(t)$, we take a closer look at the virtual motive of Q_L^n.

7.3 Reduction to the closed strata

In this section we compute the absolute virtual motive

$$[Q_L^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$$

in a different way. We need to introduce or recall some terminology.

Main characters

We fix $L = V(x,y) \subset \mathbb{A}^3$ to be the z-axis in $\mathbb{A}^3$. The "Quot to Chow" morphism takes a sheaf to its support,

$$\mathsf{s}: Q_L^n \to \operatorname{Sym}^n \mathbb{A}^3, \qquad [\mathscr{F}] \mapsto \operatorname{Supp} \mathscr{F}.$$

For motivic calculations it might often be enough to know this map is constructible. However, [70, Cor. 7.15] shows s is an actual morphism of schemes. Incidentally, by letting

$$W_L^n = \mathsf{s}^{-1}(\operatorname{Sym}^n L),$$

we get a canonical scheme structure on the closed subset $|W_L^n| \subset Q_L^n$ parametrizing subschemes $Z \subset \mathbb{A}^3$ without isolated points, cf. Definition 4.3.1. The same holds for each locally closed stratum

$$W_L^\alpha \subset W_L^n,$$

which we can now realize as the fibre of s over $\operatorname{Sym}^n_\alpha L$. We saw in (4.3.1) that W_L^α parametrizes subschemes $Z \subset \mathbb{A}^3$ whose embedded points have support distributed according to the partition α. So, if $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots r^{\alpha_r})$, a point $[Z] \in W_L^\alpha$ represents a subscheme consisting of L carrying α_i embedded points of multiplicity i, for all $i = 1,\ldots,r$.

Note that the Hilbert scheme of points $\operatorname{Hilb}^n(\mathbb{A}^3 \setminus L)$ sits inside Q_L^n, via the open immersion $J \mapsto J \cap \mathscr{I}_L$. We use a special notation for the *deep strata* in Q_L^n, as these are the most important ones: we let

$$\mathsf{W}(n) = W_L^{(n)}, \qquad \mathsf{H}(n) = \operatorname{Hilb}^n_{(n)}(\mathbb{A}^3 \setminus L).$$

They correspond to a thick *embedded* point on L and to a thick *isolated* point away from L respectively. Recall from Definition 4.3.3 that

$$F_n \subset \mathsf{W}(n)$$

parametrizes subschemes with a unique embedded point supported at the origin $0 \in L \subset \mathbb{A}^3$. Finally, to make some formulas more readable, we sometimes use the shorthand

$$X_L = \mathbb{A}^3 \setminus L \subset \mathbb{A}^3$$

for the open complement of the line.

Virtual motives

We should comment on our use of the word "virtual", and of the subscripts "relvir" and "vir". Strictly speaking, the only canonical virtual motives we have are the relative class

$$[Q_L^n]_{\mathrm{relvir}} \in \mathcal{M}_{Q_L^n}^{\hat{\mu}}$$

and its pushforward to a point, denoted $[Q_L^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$. We will, however, call a (relative) virtual motive every class obtained by pulling back $[Q_L^n]_{\mathrm{relvir}}$ along some locally closed subscheme of Q_L^n. The resulting class, relative or absolute, will inherit the relevant subscript.

Let us fix integers $0 \le j \le n$. If α (resp. β) is a partition of $n-j$ (resp. j), let

$$\mathsf{T}_{\alpha\beta} = \mathrm{Sym}_{\alpha}^{n-j}(X_L) \times \mathrm{Sym}_{\beta}^{j}(L) \subset \mathrm{Sym}^n \mathbb{A}^3.$$

We are fixing "$n-j$ points" away from L and "j points" on L, whose multiplicities are prescribed by the given partitions. We define locally closed subschemes $\mathsf{S}_{\alpha\beta} \subset Q_L^n$ via the fibre squares

$$\begin{tikzcd} \mathsf{S}_{\alpha\beta} \arrow[r, hook, "\iota_{\alpha\beta}"] \arrow[d] \arrow[dr, phantom, "\square"] & Q_L^n \arrow[d, "\mathsf{s}"] \\ \mathsf{T}_{\alpha\beta} \arrow[r, hook] & \mathrm{Sym}^n \mathbb{A}^3 \end{tikzcd} \tag{7.3.1}$$

and we note that the decomposition

$$Q_L^n = \coprod_{j=0}^{n} \coprod_{\alpha,\beta} \mathsf{S}_{\alpha\beta}$$

is nothing but a slight refinement of the stratification (4.4.1). Pushing forward (to Spec $\mathbb{C}$) the relative motives

$$[\mathsf{S}_{\alpha\beta}]_{\mathrm{relvir}} = \iota_{\alpha\beta}^{*}[Q_L^n]_{\mathrm{relvir}} \in \mathcal{M}_{\mathsf{S}_{\alpha\beta}}^{\hat{\mu}}$$

yields a decomposition

$$[Q_L^n]_{\mathrm{vir}} = \sum_{j=0}^{n} \sum_{\alpha,\beta} [\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}. \tag{7.3.2}$$

There are other important classes we need to define. Let $\mathrm{Hilb}^n(\mathbb{A}^3)_0$ be the punctual Hilbert scheme. Remembering the identifications

$$\mathsf{W}(n) = L \times F_n, \qquad \mathsf{H}(n) = X_L \times \mathrm{Hilb}^n(\mathbb{A}^3)_0,$$

we have closed immersions $F_n \subset \mathsf{W}(n)$ and $\mathrm{Hilb}^n(\mathbb{A}^3)_0 \subset \mathsf{H}(n)$ by choosing base points $0 \in L$ and $p \in X_L$. Thus we can define, again by restriction, the relative virtual motives

$$[F_n]_{\mathrm{relvir}} \in \mathcal{M}_{F_n}^{\hat{\mu}}, \qquad [\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{relvir}} \in \mathcal{M}_{\mathrm{Hilb}^n(\mathbb{A}^3)_0}^{\hat{\mu}}. \tag{7.3.3}$$

Definition 7.3.1. We let $[F_n]_{\mathrm{vir}}$ and $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ be the absolute motives in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ obtained by pushing forward to a point the relative classes (7.3.3). ♢

We denote by $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}^{\mathrm{BBS}}$ the motive defined in [7, Section 3].[1] It is obtained by restricting the relative virtual motive $[\mathrm{Hilb}^n(\mathbb{A}^3)]_{\mathrm{relvir}}$ from the full Hilbert scheme to the punctual Hilbert scheme

$$\mathrm{Hilb}^n(\mathbb{A}^3)_0 \subset \mathrm{Hilb}^n_{(n)}(\mathbb{A}^3) \subset \mathrm{Hilb}^n(\mathbb{A}^3),$$

and pushing forward to a point. Our plan is the following.

- We compute the virtual motives of the deep strata $\mathsf{W}(n)$ and $\mathsf{H}(n)$, and we show that all formulas involving absolute motives take place in the subring $\mathcal{M}_{\mathbb{C}}$ of monodromy-free classes.
- We show that the motive $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ of Definition 7.3.1, coming from the Quot scheme, agrees with $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}^{\mathrm{BBS}}$, coming from the Hilbert scheme. This is the content of Proposition 7.3.4 below.
- The absolute motives $[F_n]_{\mathrm{vir}}$ and $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ will turn out to be the most important classes, thanks to the power structure on $\mathcal{M}_{\mathbb{C}}$. They determine the virtual motive of Q_L^n (see Theorem 7.3.9 below).
- We generalize the construction of $[Q_L^n]_{\mathrm{vir}}$ to the case of an arbitrary smooth curve C in a smooth quasi-projective threefold. The induced virtual motive of Q_C^n is determined by the local one via the power structure.

7.3.1 The motives of the deep strata

Let $\mathsf{a} : Q_L^n \to \widetilde{Q}_L^n = \operatorname{Spec} \mathscr{O}(Q_L^n)$ be the affinization of the Quot scheme. The map $\mathsf{s} : Q_L^n \to \mathrm{Sym}^n \mathbb{A}^3$ induces a canonical (bijective) morphism $\widetilde{Q}_L^n \to \mathrm{Sym}^n \mathbb{A}^3$ extending s. For the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^3)$, this would be an isomorphism. Although the same is probably true for Q_L^n as well, all we need for the next result is the existence of a factorization

$$\begin{tikzcd} \mathcal{M}^{\hat{\mu}}_{Q_L^n} \arrow[r, "\mathsf{a}_!"] \arrow[dr, "\mathsf{s}_!"'] & \mathcal{M}^{\hat{\mu}}_{\widetilde{Q}_L^n} \arrow[d] \\ & \mathcal{M}^{\hat{\mu}}_{\mathrm{Sym}^n \mathbb{A}^3} \end{tikzcd} \tag{7.3.4}$$

which we certainly have by the universal property of the affinization.

Lemma 7.3.2. *The absolute motives $[\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}}$ live in the subring $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. The same is true for $[\mathsf{W}(n)]_{\mathrm{vir}}$ and $[\mathsf{H}(n)]_{\mathrm{vir}}$.*

Proof. We know by Theorem 6.2.8 that

$$\mathsf{a}_![\phi_{f_n}]_{Q_L^n} \in \mathcal{M}_{\widetilde{Q}_L^n}$$

1 The superscript was not present in Section 2.2.3, cf. (2.2.5), when we first mentioned this class.

so by (7.3.4) we have

$$\mathsf{s}_![Q_L^n]_{\mathrm{relvir}} \in \mathcal{M}_{\mathrm{Sym}^n \mathbb{A}^3}.$$

Then, exploiting the commutative diagram

$$\begin{tikzcd} & \mathcal{M}^{\hat{\mu}}_{\mathsf{S}_{\alpha\beta}} \arrow[d, "\mathsf{s}_!"'] & \mathcal{M}^{\hat{\mu}}_{Q_L^n} \arrow[l, "\iota^*_{\alpha\beta}"'] \arrow[d, "\mathsf{s}_!"] \\ \mathcal{M}^{\hat{\mu}}_{\mathbb{C}} & \mathcal{M}^{\hat{\mu}}_{\mathsf{T}_{\alpha\beta}} \arrow[l] & \mathcal{M}^{\hat{\mu}}_{\mathrm{Sym}^n \mathbb{A}^3} \arrow[l] \end{tikzcd}$$

induced by (7.3.1), one finds

$$\mathsf{s}_![\mathsf{S}_{\alpha\beta}]_{\mathrm{relvir}} \in \mathcal{M}_{\mathsf{T}_{\alpha\beta}}.$$

Pushing forward to a point yields the result for $[\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}}$. The same strategy applies for $\mathsf{W}(n)$ and $\mathsf{H}(n)$. □

We now determine the virtual motives of $\mathsf{W}(n)$ and $\mathsf{H}(n)$ explicitly. We exploit a particular group action under which the construction of $[Q_L^n]_{\mathrm{relvir}}$ is invariant. Consider the group $G = \mathrm{SL}_2 \times \mathbb{G}_a$, acting on $\mathcal{R}_n$ as follows. Writing $g = (M, \lambda) \in G$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2, \qquad \lambda \in \mathbb{G}_a,$$

we define the action

$$g \cdot (X, Y, Z, v, w) = (aX + bY, cX + dY, Z + \lambda \cdot \mathrm{Id}, av + bw, cv + dw).$$

It is easy to see that both U_n and T_n are invariant under this action, and moreover this action commutes with the action of GL_n on these spaces. The trace potential $\mathsf{W}_n : U_n \to \mathbb{A}^1$ is also invariant under the G-action, as one can verify by direct calculation: letting $P = (X, Y, Z, v, w) \in U_n$ and $g = (M, \lambda) \in G$ as described above, one has

$$\begin{aligned} \mathsf{W}_n(g \cdot P) &= \mathrm{Tr}((aX + bY)[cX + dY, Z + \lambda \cdot \mathrm{Id}]) \\ &= \mathrm{Tr}((aX + bY)(c[X, Z] + d[Y, Z])) \\ &= ad \cdot \mathrm{Tr}\, X[Y, Z] + bc \cdot Y[X, Z] \\ &= (ad - bc) \cdot \mathrm{Tr}\, X[Y, Z] \\ &= \mathsf{W}_n(P), \end{aligned}$$

where in the last equality we used that M has determinant 1. The G-action just described induces an action

$$\mu : G \times Q_L^n \to Q_L^n$$

on the Quot scheme. This can also be seen as the natural lift to Q_L^n of the action of G on $\mathbb{A}^3$, given by the change of coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \\ \lambda + z \end{pmatrix}. \tag{7.3.5}$$

Note that if we pick a sheaf $[\mathscr{F}] \in Q_L^n$, formula (7.3.5) says precisely what happens to $\operatorname{Supp} \mathscr{F}$ after we apply the action.

LEMMA 7.3.3. *The virtual motives* $[F_n]_{\mathrm{vir}}$ *and* $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ *live in the subring* $\mathcal{M}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$*, where one has the relations*

$$\begin{aligned} [\mathsf{W}(n)]_{\mathrm{vir}} &= \mathbb{L}\cdot [F_n]_{\mathrm{vir}} \\ [\mathsf{H}(n)]_{\mathrm{vir}} &= [X_L]\cdot[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}. \end{aligned} \tag{7.3.6}$$

PROOF. The action $\mu : G \times Q_L^n \to Q_L^n$ preserves the subschemes W_L^n and $\mathrm{Hilb}^n X_L$, as well as their deepest strata $\mathsf{W}(n)$ and $\mathsf{H}(n)$. We have a commutative diagram

$$\begin{tikzcd} F_n \arrow[d, hook] & \mathbb{G}_a \times F_n \arrow[l, "q_2"'] \arrow[r, "\sim"] \arrow[d, hook, "i"] & \mathsf{W}(n) \arrow[d, hook] \\ Q_L^n & G \times Q_L^n \arrow[l, "p_2"'] \arrow[r, "\mu"] & Q_L^n \end{tikzcd}$$

where q_2, p_2 are second projections and the isomorphism $\mathbb{G}_a \times F_n \xrightarrow{\sim} \mathsf{W}(n)$ is the one of Proposition 4.3.5. The construction of $[Q_L^n]_{\mathrm{relvir}}$ is invariant under the G-action, so we have $p_2^*[Q_L^n]_{\mathrm{relvir}} = \mu^*[Q_L^n]_{\mathrm{relvir}}$. We deduce that

$$q_2^*[F_n]_{\mathrm{relvir}} = i^* p_2^*[Q_L^n]_{\mathrm{relvir}} = i^*\mu^*[Q_L^n]_{\mathrm{relvir}} = [\mathsf{W}(n)]_{\mathrm{relvir}}.$$

Taking absolute motives, we get

$$[\mathbb{G}_a]\cdot[F_n]_{\mathrm{vir}} = [\mathsf{W}(n)]_{\mathrm{vir}},$$

proving the first identity in (7.3.6), with $[F_n]_{\mathrm{vir}}$ living in $\mathcal{M}_{\mathbb{C}}$. To get the second identity, we repeat the process with the diagram

$$\begin{tikzcd} \mathrm{Hilb}^n(\mathbb{A}^3)_0 \arrow[d, hook] & X_L \times \mathrm{Hilb}^n(\mathbb{A}^3)_0 \arrow[l, "\pi_2"'] \arrow[r, "\sim"] \arrow[d, hook, "j"] & \mathsf{H}(n) \arrow[d, hook] \\ Q_L^n & G \times Q_L^n \arrow[l, "p_2"'] \arrow[r, "\mu"] & Q_L^n \end{tikzcd}$$

where π_2, p_2 are second projections and the map j is defined as follows. Recall that $\mathrm{Hilb}^n(\mathbb{A}^3)_0$ is embedded in Q_L^n as the locus of fat points ξ supported at a given $p \in X_L$. Then j takes $(x,\xi) \mapsto (g_x,\xi)$, where $g_x \in G$ is the unique element that brings p to x, according to (7.3.5). We find

$$\pi_2^*[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{relvir}} = j^* p_2^*[Q_L^n]_{\mathrm{relvir}} = j^*\mu^*[Q_L^n]_{\mathrm{relvir}} = [\mathsf{H}(n)]_{\mathrm{relvir}}.$$

Taking absolute motives we get

$$[X_L]\cdot[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}} = [\mathsf{H}(n)]_{\mathrm{vir}},$$

as claimed, and with $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ living in $\mathcal{M}_{\mathbb{C}}$. □

7.3.2 A remark on $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$

The goal of this section is to show that the virtual motive of the punctual Hilbert scheme (see Definition 7.3.1) agrees with $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}^{\mathrm{BBS}}$, the virtual motive constructed by Behrend–Bryan–Szendrői. Consider the critical loci

$$Z(\mathrm{d}\mathsf{w}_n) = \mathrm{Hilb}^n(\mathbb{A}^3) \subset \mathrm{Hilb}_R^n, \qquad Z(\mathrm{d}f_n) = Q_L^n \subset \mathrm{Quot}_K^n.$$

If we pick a quotient $\mathscr{O}_{\mathbb{A}^3} \twoheadrightarrow \mathscr{O}_Z$ (resp. $\mathscr{I}_L \twoheadrightarrow \mathscr{F}$) and we demand that the support of $\mathscr{O}_Z$ (resp. $\mathscr{F}$) be contained in $X_L = \mathbb{A}^3 \setminus L$, we end up with open immersions

$$\iota_1 : \operatorname{Hilb}^n X_L \to \operatorname{Hilb}^n(\mathbb{A}^3), \qquad \iota_2 : \operatorname{Hilb}^n X_L \to Q_L^n.$$

In other words, $\operatorname{Hilb}^n X_L$ is naturally an open subscheme of both the Hilbert scheme of $\mathbb{A}^3$ and the Quot scheme of $\mathscr{I}_L$. Note that ι_2 can be described in ideal-theoretic terms as

$$J \mapsto J \cap \mathscr{I}_L.$$

We next show that $\operatorname{Hilb}^n X_L$ is a critical locus "in the same way" on either side.

PROPOSITION 7.3.4. *Let* $\iota_1 : \operatorname{Hilb}^n X_L \to \operatorname{Hilb}^n(\mathbb{A}^3)$ *and* $\iota_2 : \operatorname{Hilb}^n X_L \to Q_L^n$ *be the natural open immersions. Then one has*

$$\iota_1^*[\operatorname{Hilb}^n(\mathbb{A}^3)]_{\mathrm{relvir}} = \iota_2^*[Q_L^n]_{\mathrm{relvir}} \in \mathcal{M}_{\operatorname{Hilb}^n X_L}^{\hat{\mu}}.$$

PROOF. It is enough to verify the following

CLAIM. There is an open subset $i : U \subset \operatorname{Hilb}_R^n$ such that $\operatorname{Hilb}^n X_L = Z(\mathrm{d}(\mathrm{w}_n \circ i))$ and one has an open immersion $\Phi : U \to \operatorname{Quot}_K^n$ compatible with the potentials.

Granting the claim, if V were the image of Φ, we would be in the situation

$$\begin{tikzcd}
\operatorname{Hilb}^n X_L \arrow[r, hook] \arrow[d, hook, "\iota_1"'] \arrow[dr, phantom, "\square"] & U \arrow[rr, "\Phi"] \arrow[d, hook, "i"] & & V \arrow[d, hook, "j"'] \arrow[dr, phantom, "\square"] & \operatorname{Hilb}^n X_L \arrow[l, hook'] \arrow[d, hook, "\iota_2"] \\
\operatorname{Hilb}^n(\mathbb{A}^3) \arrow[r, hook] & \operatorname{Hilb}_R^n \arrow[r, "\mathrm{w}_n"] & \mathbb{A}^1 & \operatorname{Quot}_K^n \arrow[l, "f_n"'] & Q_L^n \arrow[l, hook']
\end{tikzcd}$$

where the outer squares are cartesian, Φ is an isomorphism onto V and i, j are open immersions. In particular, we would have

$$\iota_1^*[\phi_{\mathrm{w}_n}]_{\operatorname{Hilb}^n(\mathbb{A}^3)} = [\phi_{\mathrm{w}_n \circ i}]_{\operatorname{Hilb}^n X_L} = [\phi_{f_n \circ j}]_{\operatorname{Hilb}^n X_L} = \iota_2^*[\phi_{f_n}]_{Q_L^n},$$

where we use Φ as a "bridge" in the second equality. We know by Example 6.2.7 that Hilb_R^n and Quot_K^n have the same dimension $d = 2n^2 + n$, so the assertion on the full relative virtual motives follows from the last displayed equation, for

$$[\operatorname{Hilb}^n(\mathbb{A}^3)]_{\mathrm{relvir}} = -\mathbb{L}^{-d/2}[\phi_{\mathrm{w}_n}]_{\operatorname{Hilb}^n(\mathbb{A}^3)}, \qquad [Q_L^n]_{\mathrm{relvir}} = -\mathbb{L}^{-d/2}[\phi_{f_n}]_{Q_L^n}.$$

Let us now prove the claim. Let $\mathscr{R} = R_{\operatorname{Hilb}_R^n} = \mathscr{O}_{\operatorname{Hilb}_R^n}\langle x, y, z\rangle$, and consider the universal left ideal

$$\mathscr{J} \subset \mathscr{R}.$$

We also have the submodule $\mathcal{K} = K \otimes_{\mathbb{C}} \mathscr{O}_{\operatorname{Hilb}_R^n} = \mathscr{O}_{\operatorname{Hilb}_R^n}\langle x, y\rangle \subset \mathscr{R}$. The commutative polynomial ring $A = \mathbb{C}[x, y, z]$ comes with the quotient map $R \twoheadrightarrow A$ given by modding out the two-sided ideal $[R, R] \subset R$. This induces a surjection

$$\mathscr{R} \twoheadrightarrow \mathscr{A} = A \otimes_{\mathbb{C}} \mathscr{O}_{\operatorname{Hilb}_R^n}$$

and we let $\overline{\mathcal{J}}$ and $\overline{\mathcal{K}}$ be the images of the corresponding submodules of $\mathcal{R}$. We then consider the ringed space $(\mathrm{Hilb}^n_R, \mathcal{A})$ and the natural $\mathcal{A}$-linear inclusion

$$\eta : \overline{\mathcal{K}} + \overline{\mathcal{J}} \hookrightarrow \mathcal{A}.$$

By $\mathcal{A}$-linearity of η, and the fact that $\mathcal{A}$ is of finite type as a module over itself, the locus where η is onto is open by an application of [73, Tag 01B4, Lemma 17.9.4]. We let $U \subset \mathrm{Hilb}^n_R$ be this open subset. Note that U captures precisely the geometric condition we are after, namely that the zero-dimensional subscheme defined by $\overline{J} \subset A$ is disjoint from the line $x = y = 0$. Thus $U \cap \mathrm{Hilb}^n(\mathbb{A}^3) = \mathrm{Hilb}^n X_L$. The ideal theoretic description of our non-commutative spaces (cf. Section 6.2.1) makes it immediate to define a morphism

$$\Phi : U \to \mathrm{Quot}^n_K, \qquad J \mapsto K \cap J.$$

Note that this does land in Quot^n_K, as $K/(K \cap J) = (K+J)/J = R/J = \mathbb{C}^n$. The morphism Φ is a bijection onto its image. Indeed, $K \cap J = K \cap J'$ implies $R/J = R/J'$, hence $J = J'$. Furthermore, the image $V = \Phi(U) \subset \mathrm{Quot}^n_K$ is open. To see this, one may use that V is constructible (by Chevalley's theorem) and irreducible (because U is irreducible, being an open subscheme of an irreducible scheme). So V is closed in an open subset of Quot^n_K. But it has the same dimension as Quot^n_K, so V is open. Now $\Phi : U \to V$ is a bijective morphism of smooth schemes, so by Zariski main theorem it must be an isomorphism. We then have an open immersion $\Phi : U \to \mathrm{Quot}^n_K$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Phi"] \arrow[d, hook, "i"'] & \mathrm{Quot}^n_K \arrow[d, "f_n"] \\ \mathrm{Hilb}^n_R \arrow[r, "\mathrm{w}_n"] & \mathbb{A}^1 \end{tikzcd}$$

which brings us in the wanted situation. □

COROLLARY 7.3.5. *We have* $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}} = [\mathrm{Hilb}^n(\mathbb{A}^3)_0]^{\mathrm{BBS}}_{\mathrm{vir}}$ *in* $\mathcal{M}_{\mathbb{C}}$*.*

PROOF. It is enough to restrict the identity of Proposition 7.3.4 further to a slice $\mathrm{Hilb}^n(\mathbb{A}^3)_0 \subset \mathrm{Hilb}^n_{(n)} X_L$ inside $\mathrm{Hilb}^n X_L$. □

7.3.3 Stratification: computing the motive $[\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}}$

The goal of this section is to compute the virtual motive of Q^n_L by determining the motives $[\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}}$ and using (7.3.2). We exploit a stratification technique we already used in Section 4.4.2, again along the same lines of [9, Section 4]. Fix integers $0 \le j \le n$ and two partitions

$$\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots r^{\alpha_r}) \vdash n - j, \qquad \beta = (1^{\beta_1} \cdots k^{\beta_k} \cdots s^{\beta_s}) \vdash j.$$

We let G_α and G_β denote, as usual, the respective automorphism groups.

Isolated points

Let $D_\alpha \subset \prod_i \mathrm{Hilb}^i(X_L)^{\alpha_i}$ be the open subscheme parametrizing finite subschemes with disjoint support. Let U_α be the image of the étale map $D_\alpha \to \mathrm{Hilb}^{n-j} X_L$ given by "taking the union". The open subscheme

$$V_\alpha = \prod_i \mathsf{H}(i)^{\alpha_i} \setminus \Delta \subset \prod_i \mathsf{H}(i)^{\alpha_i}$$

fits in the cartesian diagram

$$\begin{tikzcd} V_\alpha \arrow[r, hook] \arrow[d, "q_\alpha"'] \arrow[dr, phantom, "\square"] & D_\alpha \arrow[r, hook, "\text{open}"] \arrow[d, "\text{ét}"] & \prod_i \mathrm{Hilb}^i(X_L)^{\alpha_i} \\ \mathrm{Hilb}^{n-j}_\alpha X_L \arrow[r, hook] & U_\alpha \arrow[r, hook, "\text{open}"] & \mathrm{Hilb}^{n-j} X_L \end{tikzcd}$$

where the Galois cover q_α is the (free) quotient by G_α. Moreover, the product of Hilbert–Chow morphisms (each restricted to the deep stratum) gives a trivial fibration

$$\mathsf{p}_\alpha : V_\alpha \to \prod_i X_L^{\alpha_i} \setminus \Delta = B_\alpha \tag{7.3.7}$$

with fiber $\prod_i \mathrm{Hilb}^i(\mathbb{A}^3)_0^{\alpha_i}$.

Remark 7.3.6. The above diagram makes sense for all threefolds Y [9, Lemma 4.10]. Note that the stratum $\mathrm{Hilb}^k_\alpha Y$ is *not* equal to the whole U_α, it is just a closed subscheme. This is because not all tuples of subschemes upstairs are themselves "clusters". For instance, consider $k = 5$ and $\alpha = (1^1 2^2)$. Then one can pick 5 distinct points $p_1, \dots, p_5 \in Y$ and form the subschemes Z_1, Z_2 and Z_3 consisting of p_1, $\{p_2, p_3\}$ and $\{p_4, p_5\}$ respectively. Then $(Z_1, Z_2, Z_3) \in D_\alpha$ but its image in U_α does not lie in $\mathrm{Hilb}^5_\alpha Y$. ♢

Embedded points

Let $D_\beta \subset \prod_k (W_L^k)^{\beta_k}$ be the open subset parametrizing subschemes with disjoint (zero-dimensional) support. Let U_β be the image of the étale map $D_\beta \to W_L^j$. The open subscheme

$$V_\beta = \prod_k \mathsf{W}(k)^{\beta_k} \setminus \Delta \subset \prod_k \mathsf{W}(k)^{\beta_k}$$

fits in the cartesian diagram

$$\begin{tikzcd} V_\beta \arrow[r, hook] \arrow[d, "q_\beta"'] \arrow[dr, phantom, "\square"] & D_\beta \arrow[r, hook, "\text{open}"] \arrow[d, "\text{ét}"] & \prod_k (W_L^k)^{\beta_k} \\ W_L^\beta \arrow[r, hook] & U_\beta \arrow[r, hook, "\text{open}"] & W_L^j \end{tikzcd}$$

where the Galois cover q_β is the (free) quotient by G_β. Moreover, by Proposition 4.3.5, p. 42, we have a trivial fibration

$$\mathsf{p}_\beta : V_\beta \to \prod_k L^{\beta_k} \setminus \Delta = B_\beta \tag{7.3.8}$$

with fiber $\prod_k F_k^{\beta_k}$.

Putting it all together

We now combine the two previous paragraphs to study the $(G_\alpha \times G_\beta)$-cover

$$\begin{tikzcd} & V_\alpha \times V_\beta \arrow[d, "q_{\alpha\beta}"] & \\ \mathrm{Hilb}_\alpha^{n-j}(X_L) \times W_L^\beta \arrow[r, equal] & \mathsf{S}_{\alpha\beta} \arrow[r, hook] & Q_L^n \end{tikzcd}$$

whose meaning is, roughly speaking, that the only difference between $V_\alpha \times V_\beta$ and $\mathsf{S}_{\alpha\beta}$ is the *labeling* of the supporting points: upstairs, inside the product of the punctual strata, we have ordered tuples of clusters which may happen to have the same length, but downstairs inside Q_L^n the ordering is not present any more, and this ambiguity is killed by the automorphism group of the partitions. We now describe the covering map $q_{\alpha\beta}$ explicitly in terms of commuting matrices. A point $(\xi,\eta) \in V_\alpha \times V_\beta$ can be described as follows:

- A point $\xi \in V_\alpha$ consists of the following. For every $i = 1,\ldots,r$, one has α_i tuples (A_i, B_i, C_i, v_i) where the matrices are endomorphisms of a vector space $\mathbb{C}^i$ and v_i is a cyclic vector. As we are representing a point in a *punctual* Hilbert scheme, all three matrices have a *unique* eigenvalue; we can choose representatives so that they are all upper triangular (as they pairwise commute), so in this form the unique eigenvalue of each matrix will be displayed on the diagonal. Note, however, that either A_i or B_i will be invertible, as the support of the subscheme avoids the line $L \subset \mathbb{A}^3$ given by $x = y = 0$. This means that we can equally represent the above point as a tuple $(A_i, B_i, C_i, v_i, w_i)$ including one more vector, determined as $w_i = A_i^{-1} B_i \cdot v_i$ if, say, A_i is invertible. It is no surprise that this interpretation is actually available, as $\mathrm{Hilb}^n(\mathbb{A}^3)$ and Q_L^n agree when we restrict the support to $\mathbb{A}^3 \setminus L$. To sum up, a point $\xi \in V_\alpha$ is specified by α_i tuples $(A_i, B_i, C_i, v_i, w_i)$, such that $A_i \cdot w_i = B_i \cdot v_i$, each determining a point

  $$p_i = (\lambda_{A_i}, \lambda_{B_i}, \lambda_{C_i}) \in X_L = \mathbb{A}^3 \setminus L.$$

  Finally, the disjoint support condition says that $p_i \neq p_j$ for $i \neq j$.

- A point $\eta \in V_\beta$ is described similarly. For each $k = 1,\ldots,s$, one has β_k tuples $(X_k, Y_k, Z_k, x_k, y_k)$ where the matrices are endomorphisms of $\mathbb{C}^k$ and still subject to $X_k \cdot y_k = Y_k \cdot x_k$. The same conditions regarding spanning $\mathbb{C}^k$, unique eigenvalues and disjoint support hold (of course the support is now confined on L).

The covering map $q_{\alpha\beta}$ is the direct sum; more precisely, we have

$$q_{\alpha\beta}(\xi,\eta) = (A, B, C, a, b) \in \mathsf{S}_{\alpha\beta},$$

where $A = \bigoplus_i A_i \oplus \bigoplus_k X_k$, $a = \bigoplus_i v_i \oplus \bigoplus_k x_k$ and B, C and b are defined similarly. By the disjoint support condition, the vectors obtained retain the spanning property with respect to the action of monomials in A, B and C. Moreover, the “linearity condition” $A\cdot b = B\cdot a$ is preserved. Since the matrices A, B and C are block-diagonal, we can decompose the potential f_n as

$$\operatorname{Tr} A[B,C] = \sum_i \operatorname{Tr} A_i[B_i,C_i] + \sum_k \operatorname{Tr} X_k[Y_k,Z_k]. \tag{7.3.9}$$

The next result computes the pushforward to a point of the relative class

$$\zeta_{\alpha\beta} = q_{\alpha\beta}^*[\mathsf{S}_{\alpha\beta}]_{\mathrm{relvir}} \in \mathcal{M}^{\hat\mu}_{V_\alpha\times V_\beta}.$$

The result is a $G_\alpha\times G_\beta$-equivariant motive, and applying the quotient map $\pi_{G_\alpha\times G_\beta}$ to it gives precisely $[\mathsf{S}_{\alpha\beta}]_{\mathrm{vir}}$. Recall the quasi-affine varieties

$$B_\alpha = \prod_i X_L^{\alpha_i}\setminus\Delta, \qquad B_\beta = \prod_k L^{\beta_k}\setminus\Delta$$

from the previous paragraphs.

LEMMA 7.3.7. *The pushforward of $\zeta_{\alpha\beta}$ to a point is the class*

$$\left([B_\alpha]\cdot\prod_i[\operatorname{Hilb}^i(\mathbb{A}^3)_0]_{\mathrm{vir}}^{\alpha_i}\right)\cdot\left([B_\beta]\cdot\prod_k[F_k]_{\mathrm{vir}}^{\beta_k}\right)\in\mathcal{M}_{\mathbb{C}}. \tag{7.3.10}$$

Before proving the lemma, we make an observation. Fix two schemes X_1 and X_2 and pick equivariant classes $\xi_i\in\mathcal{M}^{\hat\mu}_{X_i}$. Form the fibre product

$$\begin{array}{ccc} X_1\times X_2 & \xrightarrow{p_1} & X_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle c_1} \\ X_2 & \xrightarrow[c_2]{} & \operatorname{Spec}\mathbb{C} \end{array}$$

and let $c : X_1\times X_2 \to \operatorname{Spec}\mathbb{C}$ be the structure morphism. Then[2] one has

$$c_!(p_1^*\xi_1\star p_2^*\xi_2) = c_{1!}(\xi_1)\star c_{2!}(\xi_2)\in\mathcal{M}^{\hat\mu}_{\mathbb{C}}. \tag{7.3.11}$$

PROOF OF LEMMA 7.3.7. Applying motivic Thom–Sebastiani (Theorem 2.1.17, p. 14) to the decomposition (7.3.9), we can write $\zeta_{\alpha\beta}$ as a product of the form

$$\zeta_{\alpha\beta} = \cdots\star[\mathsf{H}(i)]_{\mathrm{relvir}}\big|_{V_\alpha\times V_\beta}\star[\mathsf{W}(k)]_{\mathrm{relvir}}\big|_{V_\alpha\times V_\beta}\star\cdots \tag{7.3.12}$$

where $\star$ is the convolution product on $\mathcal{M}^{\hat\mu}_{V_\alpha\times V_\beta}$ and the restriction is via the projection maps from

$$V_\alpha\times V_\beta\subset\prod_{i,k}\mathsf{H}(i)^{\alpha_i}\times\mathsf{W}(k)^{\beta_k}.$$

Let $\mathsf{p}=\mathsf{p}_\alpha\times\mathsf{p}_\beta$ be the product of the trivial fibrations (7.3.7) and (7.3.8) living over B_α and B_β. During the proof of Lemma 7.3.3 we showed

$$[\mathsf{H}(i)]_{\mathrm{relvir}} = \pi_2^*[\operatorname{Hilb}^i(\mathbb{A}^3)_0]_{\mathrm{relvir}}, \qquad [\mathsf{W}(k)]_{\mathrm{relvir}} = q_2^*[F_k]_{\mathrm{relvir}}$$

2 We thank Ben Davison for showing us a proof of this fact.

where $\pi_2 : \mathsf{H}(i) \to \mathrm{Hilb}^i(\mathbb{A}^3)_0$ and $q_2 : \mathsf{W}(k) \to F_k$ are the projections. Now we form the fibre diagram

$$\begin{tikzcd} V_\alpha \times V_\beta \arrow[r, hook] \arrow[d, "\mathsf{p}"'] \arrow[rr, bend left, "\mathsf{g}"] \arrow[dr, phantom, "\square"] & \prod_{i,k} \mathsf{H}(i)^{\alpha_i} \times \mathsf{W}(k)^{\beta_k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{i,k} \mathrm{Hilb}^i(\mathbb{A}^3)_0^{\alpha_i} \times F_k^{\beta_k} \arrow[d, "\mathsf{c}"] \\ B_\alpha \times B_\beta \arrow[r, hook] \arrow[rr, bend right, "\mathsf{i}"'] & \prod_{i,k} X_L^{\alpha_i} \times L^{\beta_k} \arrow[r] & \operatorname{Spec} \mathbb{C} \end{tikzcd}$$

and we use the projections

$$\begin{tikzcd} \prod_{i,k} \mathrm{Hilb}^i(\mathbb{A}^3)_0^{\alpha_i} \times F_k^{\beta_k} \arrow[r, "\mathsf{h}_i"] \arrow[d, "\mathsf{f}_k"'] & \mathrm{Hilb}^i(\mathbb{A}^3)_0 \\ F_k & \end{tikzcd}$$

to write each product in (7.3.12) as the pullback along g of the product motive

$$\mathsf{h}_i^*\big[\mathrm{Hilb}^i(\mathbb{A}^3)_0\big]_{\mathrm{relvir}} \star \mathsf{f}_k^*\big[F_k\big]_{\mathrm{relvir}}.$$

Then the class we want to compute is

$$\begin{aligned} \mathsf{i}_!\mathsf{p}_!\zeta_{\alpha\beta} &= \mathsf{i}_!\mathsf{p}_!\mathsf{g}^*\big(\cdots \star \mathsf{h}_i^*\big[\mathrm{Hilb}^i(\mathbb{A}^3)_0\big]_{\mathrm{relvir}} \star \mathsf{f}_k^*\big[F_k\big]_{\mathrm{relvir}} \star \cdots\big) \\ &= \mathsf{i}_!\mathsf{i}^*\mathsf{c}_!\big(\cdots \star \mathsf{h}_i^*\big[\mathrm{Hilb}^i(\mathbb{A}^3)_0\big]_{\mathrm{relvir}} \star \mathsf{f}_k^*\big[F_k\big]_{\mathrm{relvir}} \star \cdots\big) \\ &= \big[B_\alpha \times B_\beta\big] \cdot \mathsf{c}_!\big(\cdots \star \mathsf{h}_i^*\big[\mathrm{Hilb}^i(\mathbb{A}^3)_0\big]_{\mathrm{relvir}} \star \mathsf{f}_k^*\big[F_k\big]_{\mathrm{relvir}} \star \cdots\big), \end{aligned}$$

and the claimed formula follows from (7.3.11), after converting $\star$ to the ordinary product in $\mathcal{M}_{\mathbb{C}}$ thanks to Lemma 7.3.3. $\square$

Definition 7.3.8. Let β be a partition of j. We define the classes

$$\big[W_L^\beta\big]_{\mathrm{vir}} = \pi_{G_\beta}\left(\big[B_\beta\big] \cdot \prod_k \big[F_k\big]_{\mathrm{vir}}^{\beta_k}\right), \qquad \big[W_L^j\big]_{\mathrm{vir}} = \sum_{\beta \vdash j} \big[W_L^\beta\big]_{\mathrm{vir}}$$

in the ring of motivic weights $\mathcal{M}_{\mathbb{C}}$. $\diamondsuit$

The virtual motive of any stratum of the Hilbert scheme of points on an arbitrary threefold was defined in [7, Definition. 4.1], entirely in terms of the virtual motive $[\mathrm{Hilb}^i(\mathbb{A}^3)_0]_{\mathrm{vir}}^{\mathrm{BBS}}$ (and of the given threefold). The full motive is defined to be

$$\big[\mathrm{Hilb}^k Y\big]_{\mathrm{vir}} = \sum_{\alpha \vdash k} \big[\mathrm{Hilb}_\alpha^k Y\big]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}. \tag{7.3.13}$$

THEOREM 7.3.9. *In $\mathcal{M}_{\mathbb{C}}$ we have the relation*

$$\big[Q_L^n\big]_{\mathrm{vir}} = \sum_{j=0}^n \big[\mathrm{Hilb}^{n-j} X_L\big]_{\mathrm{vir}} \cdot \big[W_L^j\big]_{\mathrm{vir}}. \tag{7.3.14}$$

PROOF. Consider the threefold X_L and the stratum $\mathrm{Hilb}^{n-j}_\alpha X_L$ corresponding to $\alpha \vdash n-j$. Using that $[\mathrm{Hilb}^i(\mathbb{A}^3)_0]_{\mathrm{vir}} = [\mathrm{Hilb}^i(\mathbb{A}^3)_0]^{\mathrm{BBS}}_{\mathrm{vir}}$ (cf. Corollary 7.3.5), the definition [7, Definition. 4.1] mentioned above reads

$$\left[\mathrm{Hilb}^{n-j}_\alpha X_L\right]_{\mathrm{vir}} = \pi_{G_\alpha}\left(\left[B_\alpha\right]\cdot \prod_i \left[\mathrm{Hilb}^i(\mathbb{A}^3)_0\right]^{\alpha_i}_{\mathrm{vir}}\right).$$

The motive (7.3.10) computed in Lemma 7.3.7 defines a class in the equivariant motivic ring

$$\widetilde{\mathcal{M}}^{G_\alpha\times G_\beta}_{\mathbb{C}}$$

by Lemma 2.1.7. Taking its image under the quotient map $\pi_{G_\alpha\times G_\beta}$, defined in (2.1.5), yields

$$\left[\mathsf{S}_{\alpha\beta}\right]_{\mathrm{vir}} = \left[\mathrm{Hilb}^{n-j}_\alpha X_L\right]_{\mathrm{vir}}\cdot \left[W^\beta_L\right]_{\mathrm{vir}}. \tag{7.3.15}$$

Combining (7.3.13) with the definition of $[W^j_L]_{\mathrm{vir}}$, the decomposition (7.3.2) finally proves the result by summing over j, α and β. □

Let us define the generating function

$$\mathsf{F}(t) = \sum_{n\geq 0}\left[F_n\right]_{\mathrm{vir}} t^n \in \mathcal{M}_{\mathbb{C}}[\![t]\!].$$

We then have the following.

COROLLARY 7.3.10. *The motivic partition function* Z *of the Quot scheme can be written as*

$$\mathsf{Z}(t) = \mathsf{Z}_{\mathbb{A}^3,0}(t)^{\mathbb{L}^3-\mathbb{L}}\cdot \mathsf{F}(t)^{\mathbb{L}}.$$

PROOF. Using the power structure on the ring of motivic weights, (2.2.2) gives

$$\sum_{n\geq 0}\left[W^n_L\right]_{\mathrm{vir}} t^n = \mathsf{F}(t)^{\mathbb{L}}. \tag{7.3.16}$$

By (7.3.14) we can write

$$\mathsf{Z}(t) = \mathsf{Z}_{\mathbb{A}^3\setminus L}(t)\cdot \mathsf{F}(t)^{\mathbb{L}}.$$

The result now follows from Theorem 2.2.4 applied to $\mathbb{A}^3\setminus L$. □

From Corollary 7.3.10 we indeed see that the virtual motives of the deepest strata, $[\mathrm{Hilb}^n(\mathbb{A}^3)_0]_{\mathrm{vir}}$ and $[F_n]_{\mathrm{vir}}$, determine the motivic partition function Z of the Quot scheme. It would be nice to have a closed formula for $[F_n]_{\mathrm{vir}}$.

Arbitrary curves

Let Y be a smooth quasi-projective threefold, $C\subset Y$ a smooth curve. Recall the Quot scheme $Q^n_C = \mathrm{Quot}_n(\mathscr{I}_C)$, the main character of Chapter 4.

Definition 7.3.11. Let $j\geq 0$ be an integer. We call the motivic class

$$\left[W^j_C\right]_{\mathrm{vir}} = \sum_{\beta\vdash j}\pi_{G_\beta}\left(\left[\prod_k C^{\beta_k}\setminus\Delta\right]\cdot \prod_k \left[F_k\right]^{\beta_k}_{\mathrm{vir}}\right)\in \mathcal{M}_{\mathbb{C}}$$

the *virtual motivic contribution* of $W^j_C \subset Q^j_C$. ◇

Definition 7.3.12. Let $n \geq 0$ be an integer. We define the motivic class

$$[Q_C^n]_{\mathrm{vir}} = \sum_{j=0}^{n} [\mathrm{Hilb}^{n-j}(Y \setminus C)]_{\mathrm{vir}} \cdot [W_C^j]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$$

and the generating function

$$\mathsf{Z}_{C/Y}(t) = \sum_{n \geq 0} [Q_C^n]_{\mathrm{vir}} t^n. \qquad \diamondsuit$$

Again, the subscript "vir" has nothing to do with the canonical virtual motive of a critical locus. In fact, we have not computed the weighted Euler characteristic of W_C^j, so $[W_C^j]_{\mathrm{vir}}$ need not be a virtual motive. However, we will show below that $[Q_C^n]_{\mathrm{vir}}$ is a virtual motive. Finally, the notation $\mathsf{Z}_{C/Y}$ reminds us that the classes defined above are not intrinsic to C, but depend on its embedding into Y, as Q_C^n does. Note that $\mathsf{Z}_{L/\mathbb{A}^3} = \mathsf{Z}$ by Theorem 7.3.9.

Lemma 7.3.13. *We have $\chi([F_n]_{\mathrm{vir}}) = (-1)^n \chi(F_n)$ for all $n \geq 0$.*

Proof. This can be proven by induction, the case $n = 0$ being clear. Combining Theorem 4.4.1 with the fact that $[Q_L^n]_{\mathrm{vir}}$ is a virtual motive, we find

$$(-1)^n \chi(Q_L^n) = \chi_{\mathrm{vir}}(Q_L^n) = \chi\left([Q_L^n]_{\mathrm{vir}}\right).$$

Moreover, we know by Theorem 7.3.9 that

$$[Q_L^n]_{\mathrm{vir}} = \sum_{j=0}^{n} [\mathrm{Hilb}^{n-j} X_L]_{\mathrm{vir}} \cdot [W_L^j]_{\mathrm{vir}}.$$

Taking the Euler characteristic of the right hand side, and using the previous relation, it is easy to apply the inductive step. $\square$

Theorem 7.3.14. *The class $[Q_C^n]_{\mathrm{vir}}$ is a virtual motive for Q_C^n, and*

$$\mathsf{Z}_{C/Y}(t) = \mathsf{Z}_{\mathbb{A}^3,0}(t)^{[Y \setminus C]} \cdot \mathsf{F}(t)^{[C]}.$$

Proof. By Lemma 7.3.13, we have

$$\chi\left([W_C^j]_{\mathrm{vir}}\right) = \sum_{\beta \vdash j} \chi(\mathrm{Sym}_\beta^j C) \cdot \prod_k (-1)^{k\beta_k} \chi(F_k)^{\beta_k} = (-1)^j \chi(W_C^j),$$

so that $\chi([Q_C^n]_{\mathrm{vir}}) = (-1)^n \chi(Q_C^n)$. Then Theorem 4.4.1 makes $[Q_C^n]_{\mathrm{vir}}$ into a virtual motive for Q_C^n. The assertion on $\mathsf{Z}_{C/Y}$ follows by the very definition of $[Q_C^n]_{\mathrm{vir}}$ along with Theorem 2.2.4 (applied to $Y \setminus C$), and noting that

$$\sum_{j \geq 0} [W_C^j]_{\mathrm{vir}} t^j = \mathsf{F}(t)^{[C]}$$

by formula (2.2.2) defining the power structure. $\square$

Corollary 7.3.15. *Let Y be a projective Calabi–Yau threefold, $C \subset Y$ a smooth curve with $n_{g,C} = 1$. Then*

$$\chi\left([Q_C^n]_{\mathrm{vir}}\right) = \mathsf{DT}_{n,C}.$$

PROOF. Combining Theorem 7.3.14 with the local DT/PT correspondence (Theorem 5.1.1, p. 61), one finds $\chi([Q_C^n]_{\mathrm{vir}}) = \chi_{\mathrm{vir}}(Q_C^n) = \mathsf{DT}_{n,C}$. □

In particular, $[Q_C^n]_{\mathrm{vir}}$ can be seen as a "local" motivic DT invariant of Y at C. When C is rigid, for example, $\mathsf{DT}_{n,C}$ is really the degree of the virtual fundamental class

$$[Q_C^n]^{\mathrm{vir}} \in A_0(Q_C^n),$$

naturally defined on the *connected component*

$$Q_C^n = I_n(Y, C) \subset I_{1-g+n}(Y, [C])$$

of the full moduli space. So its refinement $[Q_C^n]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}}$ is a motivic DT invariant in the strong sense of Definition 2.1.8.

Remark 7.3.16. In [77, Example 5.7] one can find an example of a cohomological DT invariant in the projective case. We are not aware of other examples of motivic DT invariants for projective Calabi–Yau threefolds, in a setting where the moduli space parametrizes *curves and points*. Of course, without a curve in the picture, we do have the virtual motive $[\mathrm{Hilb}^n Y]_{\mathrm{vir}}$ constructed in [7] for arbitrary threefolds, and if Y is an *open* Calabi–Yau there are plenty of examples, see for instance [51, 52, 23, 55]. ◊

8 TOWARDS A MOTIVIC DT/PT CORRESPONDENCE

8.1 Introduction

In this chapter we conjecture an explicit formula for the motivic partition function Z of the Quot schemes Q_L^n. The formula is

$$\mathsf{Z} = \mathsf{Z}_{\mathbb{A}^3} \cdot \mathsf{Z}_L, \tag{8.1.1}$$

where Z_X denotes the generating function

$$\mathsf{Z}_X(t) = \sum_{n\geq 0} \left[\operatorname{Hilb}^n X\right]_{\mathrm{vir}} t^n.$$

Of course, this is only defined if $\dim X \leq 3$, and $\mathsf{Z}_{\mathbb{A}^3}$ is the partition function studied in [7]. It encodes the 0-dimensional motivic DT theory of $\mathbb{A}^3$ and can be thought of the "point contribution" to Z. The other factor is the geometric series

$$\mathsf{Z}_L(t) = \sum_{n\geq 0} \left[\operatorname{Sym}^n L\right]_{\mathrm{vir}} t^n = (1 - t\mathbb{L}^{1/2})^{-1},$$

the most natural motivic refinement of the "stable pair moduli space" $\operatorname{Sym}^n L$. It should be interpreted as the "curve contribution" to Z. The conjectured identity (8.1.1) immediately generates (via the power structure) analogous formulas for the partition functions $\mathsf{Z}_{C/Y}$ of Definition 7.3.12, where C is any smooth curve inside a smooth quasi-projective threefold Y. The predicted formula reads

$$\mathsf{Z}_{C/Y} = \mathsf{Z}_Y \cdot \mathsf{Z}_C. \tag{8.1.2}$$

When Y is a smooth *projective* threefold and $C \subset Y$ is a smooth curve of genus g, formula (8.1.2) can be seen as a motivic refinement of the identity

$$\sum_{n\geq 0} \tilde{\chi}(I_n(Y,C))q^n = M(-q)^{\chi(Y)}(1+q)^{2g-2}$$

proved in Proposition 4.5.6, where $I_n(Y,C) = Q_C^n$. When Y is a projective Calabi–Yau threefold and C has BPS number 1, (8.1.2) refines the (numerical) DT/PT correspondence

$$\mathsf{DT}_C = \mathsf{DT}_0(Y) \cdot \mathsf{PT}_C$$

proved in Chapter 5. Therefore (8.1.2) might be called a *motivic wall-crossing formula* at $C \subset Y$.

We show that (8.1.1) holds to order up to 4. To compare n-th coefficients, it is essential to understand the structure of the stack of coherent modules of length n over the ring $\mathbb{C}[x,y]$. In a joint work with Riccardo Moschetti [53], we carried out the complete classification of such modules for $n \leq 4$. We use some of the results in *loc. cit.*, but not the whole classification is needed for the sake of verifying the proposed formula.

8.2 A conjectural formula for $\mathsf{Z}(t)$

By Corollary 7.3.10, and exploiting the properties of the power structure along with Theorem 2.2.4, we may write

$$\mathsf{Z}(t) = \mathsf{Z}_{\mathbb{A}^3}(t) \cdot \frac{\mathsf{F}(t)^{\mathbb{L}}}{\mathsf{Z}_{\mathbb{A}^3,0}(t)^{\mathbb{L}}}.$$

Unfortunately we do not have a direct strategy to compute F, but we just established that $\mathsf{Z}_{\mathbb{A}^3}$ is a factor of Z. It is reasonable to believe this factor to account for the whole "0-dimensional contribution" to Z, so we need to interpret

$$\frac{\mathsf{F}(t)^{\mathbb{L}}}{\mathsf{Z}_{\mathbb{A}^3,0}(t)^{\mathbb{L}}}$$

as the "curve contribution". We next conjecture the latter fraction to equal the generating function

$$\mathsf{Z}_L(t) = \sum_{n\geq 0} \left[\operatorname{Sym}^n L\right]_{\mathrm{vir}} t^n.$$

Note that $[\operatorname{Sym}^n L]_{\mathrm{vir}} = \mathbb{L}^{-n/2}[\mathbb{A}^n] = \mathbb{L}^{n/2}$ by Example 2.1.14, p. 13, thus

$$\mathsf{Z}_L(t) = (1 - t\mathbb{L}^{1/2})^{-1}$$

is a simple geometric series.

Conjecture 2 ("Motivic wall-crossing")**.** In $\mathcal{M}_{\mathbb{C}}[\![t]\!]$, one has the identity

$$\mathsf{Z}(t) = \mathsf{Z}_{\mathbb{A}^3}(t) \cdot \mathsf{Z}_L(t). \tag{8.2.1}$$ ♣

8.2.1 Equivalent formulations

Conjecture 2 is clearly equivalent to the expression

$$\left[Q_L^n\right]_{\mathrm{vir}} = \sum_{k=0}^{n} \left[\operatorname{Hilb}^{n-k}(\mathbb{A}^3)\right]_{\mathrm{vir}} \cdot \mathbb{L}^{k/2}, \tag{8.2.2}$$

where we should interpret $\mathbb{L}^{k/2} = [\operatorname{Sym}^k L]_{\mathrm{vir}}$. We already know that the virtual motives of Q_L^n and of F_n determine each other (cf. Theorem 7.3.9, p. 96): when written in the form $\mathsf{F}(t)^{\mathbb{L}} = \mathsf{Z}_{\mathbb{A}^3,0}(t)^{\mathbb{L}} \cdot \mathsf{Z}_L(t)$ the conjecture predicts

$$\left[F_n\right]_{\mathrm{vir}} = \sum_{k=0}^{n} \left[\operatorname{Hilb}^{n-k}(\mathbb{A}^3)_0\right]_{\mathrm{vir}} \cdot \mathbb{L}^{-k/2}. \tag{8.2.3}$$

On the other hand, we may use formula (2.3.11) to express $\mathsf{Z}_{\mathbb{A}^3}(t)$ as the fraction $\mathsf{C}(t\mathbb{L}^{1/2})/\mathsf{C}(t\mathbb{L}^{-1/2})$, and then the relation $\mathsf{Z}(t)=\mathsf{E}(t\mathbb{L}^{-1/2})/\mathsf{C}(t\mathbb{L}^{-1/2})$ of Theorem 7.2.1 says that

$$\mathsf{E}(t\mathbb{L}^{-1/2})=\mathsf{C}(t\mathbb{L}^{1/2})\cdot\frac{\mathsf{F}(t)^{\mathbb{L}}}{\mathsf{Z}_{\mathbb{A}^3,0}(t)^{\mathbb{L}}}.$$

Then Conjecture 2 can be rephrased as $\mathsf{E}(t)=\mathsf{C}(t\mathbb{L})\cdot\mathsf{Z}_L(t\mathbb{L}^{1/2})$, that is,

$$\frac{[E_n]}{\mathsf{GL}_n}=\mathbb{L}^n\cdot\sum_{k=0}^{n}[\mathcal{C}(k)]. \tag{8.2.4}$$

Example 8.2.1. By the properties of the power structure, we deduce from Theorem 2.2.4 the expression

$$\begin{aligned}\mathsf{Z}_{\mathbb{A}^3,0}(t)&=\prod_{m\geq 1}\prod_{k=0}^{m-1}(1-\mathbb{L}^{k-1-m/2})^{-1}\\&=1+\mathbb{L}^{-3/2}t+\mathbb{L}^{-3}(1+\mathbb{L}+\mathbb{L}^2)t^2+\cdots\end{aligned}$$

For example, if $n=1$, the conjecture predicts

$$[F_1]_{\mathrm{vir}}=\mathbb{L}^{-3/2}+\mathbb{L}^{-1/2}.$$

Note that $F_1=\mathbb{P}^1$, and the above class can be interpreted as $\mathbb{L}^{-3/2}[F_1]$, where the "3" in the exponent reminds us that we are restricting the virtual motive of the smooth threefold $Q_L^1=\mathrm{Bl}_L\,\mathbb{A}^3$. Note in particular that $[F_1]_{\mathrm{vir}}\neq[\mathbb{P}^1]_{\mathrm{vir}}$, the latter being defined as $\mathbb{L}^{-1/2}(\mathbb{L}+1)$, cf. Example 2.1.14, p. 13. ♢

Remark 8.2.2. Of course, to compute F is equivalent to compute E. However, trying to prove (8.2.4) seems more approachable than proving (8.2.3), for all "virtualness" has gone away. This is why we will mainly focus on (8.2.4). ♢

8.2.2 Induced formulas for any $C\subset Y$

Let Y be a smooth quasi-projective threefold, $C\subset Y$ a smooth curve. Suppose for a moment Conjecture 2 is true. Then Theorem 7.3.14 combined with the properties of the power structure yields

$$\begin{aligned}\mathsf{Z}_{C/Y}(t)&=\mathsf{Z}_Y(t)\cdot\left(\frac{\mathsf{F}(t)}{\mathsf{Z}_{\mathbb{A}^3,0}(t)}\right)^{[C]}\\&=\mathsf{Z}_Y(t)\cdot\mathsf{Z}_L(t)^{\mathbb{L}^{-1}[C]}\\&=\mathsf{Z}_Y(t)\cdot(1-t\mathbb{L}^{-1/2})^{-[C]}\\&=\mathsf{Z}_Y(t)\cdot\sum_{n\geq 0}\mathbb{L}^{-n/2}[\mathrm{Sym}^n\,C]t^n,\end{aligned}$$

which can be rephrased as

$$\mathsf{Z}_{C/Y}=\mathsf{Z}_Y\cdot\mathsf{Z}_C. \tag{8.2.5}$$

In the projective case, (8.2.5), if true, refines the identity

$$\sum_{n\geq 0}\tilde{\chi}(Q_C^n)t^n = M(-t)^{\chi(Y)}(1+t)^{2g-2}$$

of Proposition 4.5.6. In the Calabi–Yau case, and when the BPS number of C equals 1, it refines the DT/PT correspondence

$$\mathsf{DT}_C(q) = \mathsf{DT}_0(Y,q)\cdot \mathsf{PT}_C(q)$$

of Chapter 5. So in this case we view (8.2.5) as a *motivic DT/PT correspondence* at $C \subset Y$.

8.2.3 The cases $n = 0$, 1

The conjecture in the form of equation (8.2.2) is true for $n = 0$ (trivially) and $n = 1$ (this is the content of Example 6.2.7, p. 79).

For $n = 1$, we may also want to verify equation (8.2.4) directly as follows. We have the affine quadric threefold

$$E_1 = \left\{ (A,B,a,b) \,\middle|\, A\cdot b = B\cdot a \right\} \subset \mathbb{A}^4$$

together with the map $E_1 \to C_1 = \mathbb{A}^2$ forgetting (a,b). There are two strata. The fibre over $0 \in \mathbb{A}^2$ is a copy of $\mathbb{A}^2$, while above $\mathbb{A}^2 \setminus 0$ the map is locally trivial with fibre $\mathbb{A}^1$. Hence

$$[E_1] = \mathbb{L}^2 + \mathbb{L}(\mathbb{L}^2 - 1).$$

In other words, using that $\mathcal{C}(1) = \mathbb{A}^2/\mathbb{G}_m$, we find

$$\frac{[E_1]}{\mathbb{L}-1} = \frac{\mathbb{L}^2+\mathbb{L}(\mathbb{L}^2-1)}{\mathbb{L}-1} = \mathbb{L}\frac{\mathbb{L}+\mathbb{L}^2-1}{\mathbb{L}-1} = \mathbb{L}\cdot\big(1+[\mathcal{C}(1)]\big).$$

So (8.2.4) holds for $n = 1$.

It is possible to continue and check the formula directly also for $n = 2$. However, the argument gets quite involved and is not particularly enlightening. We prefer to try another approach, which will in the end confirm the conjecture for $n \leq 4$.

8.3 Evidence for Conjecture 2

In this section we verify a few more instances of Conjecture 2. By explicit calculation, we will show the following.

PROPOSITION 8.3.1. *Conjecture 2 is true up to order* 4*. In other words, the relation*

$$[Q_L^n]_{\mathrm{vir}} = \sum_{k=0}^{n}[\mathrm{Hilb}^{n-k}(\mathbb{A}^3)]_{\mathrm{vir}}\cdot \mathbb{L}^{k/2}$$

holds if $n \leq 4$.

Let us recall the main characters. The forgetful morphism $E_n \subset C_n \times V^2 \to C_n$ is GL_n-equivariant, so it descends to the corresponding quotient stacks, and we obtain a commutative diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & \mathcal{E}(n) \arrow[d, "\pi_n"] \\ C_n \arrow[r] & \mathcal{C}(n) \end{tikzcd} \tag{8.3.1}$$

where $\mathcal{E}(n) = E_n / \mathrm{GL}_n$. Recall that $\mathcal{C}(n) = C_n / \mathrm{GL}_n$ is equivalent to the stack $\mathsf{Coh}_n(\mathbb{A}^2)$ of coherent sheaves on the plane.

Notation 8.3.1. Let $A = \mathbb{C}[x,y]$ denote the coordinate ring of $\mathbb{A}^2$, and $\mathfrak{m} = (x,y) \subset A$ the maximal ideal of the origin. Let $\mathcal{C}(n)_k \subset \mathcal{C}(n)$ be the substack parametrizing sheaves such that $\mathfrak{m}$ appears with multiplicity $n-k$ in their support. For instance, $\mathcal{C}(n)_0 \subset \mathcal{C}(n)$ is the closed substack parametrizing sheaves entirely supported at the origin. We denote by $\mathcal{E}(n)_k \subset \mathcal{E}(n)$ the pullback of $\mathcal{C}(n)_k$ along π_n. ♦

Here is our strategy:

> We will think of $\mathcal{E}(n)$ as the stack of pairs $([F],\phi)$ where $[F] \in \mathcal{C}(n)$ is a sheaf and $\phi : \mathfrak{m} \to F$ is an A-linear map. Then π_n is the morphism forgetting ϕ and retaining the sheaf $[F]$. We stratify $\mathcal{C}(n)$ by the dimension of the fibre $\mathrm{Hom}_A(\mathfrak{m}, F)$ of π_n, and then we observe (Lemma 8.3.10) that in order to verify the conjecture in its form
> $$\frac{[E_n]}{\mathsf{GL}_n} = \mathbb{L}^n \cdot \sum_{k=0}^{n} [\mathcal{C}(k)],$$
> we may very well replace π_n by its restriction $\mathcal{E}(n)_0 \to \mathcal{C}(n)_0$. In other words, we only need to pay attention to sheaves supported in one point.

8.3.1 Some technical tools

Let $\mathcal{P}(n)$ be the stack defined as follows. For a complex scheme S, let $\mathcal{P}(n)(S)$ be the groupoid of pairs $(\mathscr{F},\phi)$ where $\mathscr{F}$ is an S-flat family of coherent sheaves of finite length n on $\mathbb{A}^2_S \to S$ and ϕ is an $\mathscr{O}_{\mathbb{A}^2_S}$-linear homomorphism

$$\phi : \mathfrak{m}_S \to \mathscr{F},$$

where $\mathfrak{m}_S$ is the pullback of $\mathfrak{m}$ along the projection $p : \mathbb{A}^2_S \to \mathbb{A}^2$. Given $f : T \to S$ and two objects $\xi = (\mathscr{F},\phi)$ and $\zeta = (\mathscr{E},\psi)$ lying over T and S respectively, a morphism $\xi \to \zeta$ in $\mathcal{P}(n)$ lying over f is a commutative diagram

$$\begin{tikzcd} f^*\mathfrak{m}_S \arrow[r, "f^*\psi"] \arrow[d, equal] & f^*\mathscr{E} \arrow[d, "\alpha"] \\ \mathfrak{m}_T \arrow[r, "\phi"] & \mathscr{F} \end{tikzcd}$$

where α is an isomorphism in $\mathrm{Coh}(\mathbb{A}^2_T)$. To be more precise, by f^* we actually mean $(f\times \mathrm{id}_{\mathbb{A}^2})^*$, and by the pullback symbol we understand a *choice* of pullback for every morphism of schemes, so that the equality symbol in the diagram is the canonical isomorphism induced by this choice.

LEMMA 8.3.2. *The stack $\mathcal{E}(n)$ is equivalent to $\mathcal{P}(n)$.*

PROOF. One can identify E_n with the space P_n of triples $(A,B,\widetilde{\phi})$ where $(A,B)\in C_n$ and $\widetilde{\phi}:\mathfrak{m}\to V$ is a $\mathbb{C}$-linear map satisfying $A\cdot\widetilde{\phi}(y)=B\cdot\widetilde{\phi}(x)$. The isomorphism $E_n \xrightarrow{\sim} P_n$ is an isomorphism of GL_n-spaces, where the GL_n-action on P_n is given by $g\cdot(A,B,\widetilde{\phi})=(A^g,B^g,g\circ\widetilde{\phi})$. Taking stack quotients, we get an equivalence $\mathcal{E}(n)\xrightarrow{\sim}\mathcal{P}(n)$. □

Some arguments in the following proofs develop along the same lines of similar results in [15, Section 2].

LEMMA 8.3.3. *The stack $\mathcal{E}(n)$ is algebraic. The morphism $\pi_n:\mathcal{E}(n)\to\mathcal{C}(n)$ is representable and of finite type.*

We need the following result of Grothendieck, which we recall almost verbatim from [57, Thm. 5.8]. Let $f:X\to S$ be a projective morphism, E and F two coherent sheaves on X. Consider the functor $\mathrm{Sch}_S^{\mathrm{op}}\to\mathrm{Sets}$ sending an S-scheme $T\to S$ to the set of morphism $\mathrm{Hom}_{X_T}(E_T,F_T)$, where E_T and F_T are the pullbacks of E and F along the projection $X_T=X\times_S T\to X$. Then, if F is flat over S, the above functor is represented by a linear scheme $\mathbf{V}=\mathrm{Spec}\,\mathrm{Sym}_{\mathscr{O}_S}\mathscr{H}\to S$, where $\mathscr{H}$ is a coherent sheaf on S. We need to compactify $\mathbb{A}^2$ in order to apply this result.

PROOF OF LEMMA 8.3.3. Embed $\mathbb{A}^2\subset\mathbb{P}^2$ as the complement of the third coordinate hyperplane $x_2=0$, and form the stacks $\overline{\mathcal{C}}(n)=\mathrm{Coh}_n(\mathbb{P}^2)$ and $\overline{\mathcal{P}}(n)$. The latter parametrizes pairs (F,ϕ) such that F is a coherent sheaf of length n on $\mathbb{P}^2$ and $\phi:\overline{\mathfrak{m}}\to F$ is an $\mathscr{O}_{\mathbb{P}^2}$-linear morphism, where $\overline{\mathfrak{m}}$ is the ideal of the point $(0:0:1)\in\mathbb{P}^2$. Let $\overline{\pi}_n:\overline{\mathcal{P}}(n)\to\overline{\mathcal{C}}(n)$ be the morphism forgetting the map and retaining the sheaf, so that π_n is (up to identifying $\mathcal{P}(n)$ with $\mathcal{E}(n)$ via Lemma 8.3.2) the pullback of $\overline{\pi}_n$ along the open substack $\mathcal{C}(n)\subset\overline{\mathcal{C}}(n)$. Let S be a scheme, $S\to\overline{\mathcal{C}}(n)$ a morphism corresponding to a flat family of sheaves F parametrized by S. Let

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{P}}(n) \arrow[d, "\overline{\pi}_n"] \\
S \arrow[r] & \overline{\mathcal{C}}(n)
\end{tikzcd}
\tag{8.3.2}
$$

be the fibre product. Then P is fibred in sets, corresponding to a functor sending $T\to S$ to $\mathrm{Hom}_{\mathbb{P}^2_T}(\overline{\mathfrak{m}}_T,F_T)$. By S-flatness of F, and thanks to the result recalled above, this functor is represented by a linear *scheme* $\mathbf{V}\to S$, showing that $\overline{\pi}_n$ is representable. Taking S to be an atlas for $\overline{\mathcal{C}}(n)$ shows that $\overline{\mathcal{P}}(n)$ is algebraic. Pulling this back to the open substack $\mathcal{C}(n)\subset\overline{\mathcal{C}}(n)$ proves the result. □

Definition 8.3.4. By a *Zariski fibration* of stacks we mean a morphism $\mathscr{X} \to \mathscr{Y}$ such that the pullback along any morphism $B \to \mathscr{Y}$ from a scheme is a Zariski fibration of schemes (cf. Definition 2.1.2, p. 7). ♢

Remark 8.3.5. Note that a Zariski fibration of stacks is automatically representable, but the definition does *not* imply that $\mathscr{Y}$ has an open cover by substacks such that the pullback becomes trivial. This is why in the definition of Grothendieck group of algebraic stacks one has to add the "fibration property" as an axiom. ♢

LEMMA 8.3.6. *There is a stratification of $\mathcal{C}(n)$ by locally closed substacks*

$$\mathcal{C}(n,r) \subset \mathcal{C}(n),$$

such that their pullback under π_n is a Zariski fibration with fibre $\mathbb{C}^r$.

We need to recall another result of Grothendieck. This is [31, Théorème 7.7.6] and can also be found in [57, Thm. 5.7]. If $f : X \to S$ is a proper morphism, and E is a coherent sheaf on X that is S-flat, there exists a coherent sheaf $\mathscr{Q}_E$ on S inducing functorial isomorphisms

$$\eta_{\mathscr{M}} : f_*(E \otimes_{\mathscr{O}_S} \mathscr{M}) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{O}_S}(\mathscr{Q}_E, \mathscr{M})$$

for all quasicoherent sheaf $\mathscr{M}$ on S. The sheaf $\mathscr{Q}_E$ is unique up to a unique isomorphism, it behaves well with respect to pullback, and moreover it is locally free exactly when f is cohomologically flat in dimension zero [31, Prop. 7.8.4].

PROOF OF LEMMA 8.3.6. Let S be a scheme, $E \in \mathsf{Coh}_n(\mathbb{P}^2_S)$ a flat family of sheaves corresponding to a morphism $S \to \overline{\mathcal{C}}(n)$. The projection $f : \mathbb{P}^2_S \to S$ is cohomologically flat in dimension zero: this is true for every proper flat morphism with geometrically reduced fibres, see for instance [31, Prop. 7.8.6]. It follows from the result recalled above that the sheaf $\mathscr{Q}_E$ is locally free of finite rank. Let

$$\overline{\mathcal{C}}(n,r)(S) \subset \overline{\mathcal{C}}(n)(S)$$

be the full subcategory consisting of sheaves E such that $\mathscr{Q}_E$ is locally free of rank r. By the existence and the usual properties of the flattening stratification [57, Thm. 5.13], these subcategories are substacks and form a locally closed stratification of $\overline{\mathcal{C}}(n)$.

Consider a morphism $\mu : S \to \overline{\mathcal{C}}(n,r) \subset \overline{\mathcal{C}}(n)$. Then in the fibre square (8.3.2) one has now $P = \operatorname{Spec} \operatorname{Sym}_{\mathscr{O}_S} \mathscr{Q}_E$. Since $\mathscr{Q}_E$ is locally free of rank r, the pullback $P \to S$ of $\overline{\pi}_n$ along μ is now a geometric vector bundle over S, hence Zariski locally trivial with fibre $\mathbb{C}^r$. This shows that $\overline{\pi}_n$ becomes a Zariski fibration when pulled back to $\overline{\mathcal{C}}(n,r)$. The result for π_n follows by restricting to the open substacks $\mathcal{C}(n,r) = \overline{\mathcal{C}}(n,r) \cap \mathcal{C}(n)$. □

We now focus on modules entirely supported on $\mathfrak{m}$. Let $U \to \mathcal{C}(n)_0$ be an atlas, corresponding to a family of modules parametrized by U. The function $\mathsf{r} : U \to \mathbb{N}$ defined by

$$u \mapsto \dim_{\mathbb{C}} F_u / \mathfrak{m} \cdot F_u$$

is upper semi-continuous, so its fibre over $r \in \mathbb{N}$ defines a locally closed subset U_r of U, which we may endow with the reduced scheme structure. Its image in $\mathcal{C}(n)_0$ defines a locally closed substack

$$\mathcal{X}(n)_r \subset \mathcal{C}(n)_0,$$

and $\mathcal{C}(n)_0$ is stratified by these substacks when r ranges through 1 to n. The number r represents the minimal number of generators of our modules. Note that the motivic class of $\mathcal{X}(n)_r$ is independent upon the choice of scheme structure on U_r.

COROLLARY 8.3.7. *Let $\mathcal{X}(n)_r \subset \mathcal{C}(n)_0$ be the substack of modules, supported at $\mathfrak{m}$, that have r as minimal number of generators. Then the pullback of $\pi_n : \mathcal{E}(n)_0 \to \mathcal{C}(n)_0$ along $\mathcal{X}(n)_r$ is a Zariski fibration with fibre $\mathbb{C}^{n+r}$.*

PROOF. Let us pick a point $[F] \in \mathcal{X}(n)_r$ and an A-linear map $\phi : \mathfrak{m} \to F$. Then ϕ is determined by the images $\phi(x)$ and $\phi(y)$ of the generators, along with the relation $y \cdot \phi(x) = x \cdot \phi(y)$. However, multiplication by x and y map F into the submodule $\mathfrak{m} \cdot F$, which has length $n-r$. The identity $y \cdot \phi(x) = x \cdot \phi(y)$ then imposes $n-r$ conditions, so $\dim_{\mathbb{C}} \operatorname{Hom}_A(\mathfrak{m}, F) = 2n-(n-r) = n+r$. This shows that

$$\mathcal{X}(n)_r \subset \mathcal{C}(n, n+r),$$

and since π_n is a Zariski fibration over $\mathcal{C}(n, n+r)$ by Lemma 8.3.6, the same is true over the substack $\mathcal{X}(n)_r$. □

Note that the motivic class of $\mathcal{X}(n)_r$ makes sense in the Grothendieck ring $K_0(\mathrm{St}_{\mathbb{C}})$ by the locally closed condition, so by Corollary 8.3.7 we obtain a decomposition

$$[\mathcal{E}(n)_0] = \mathbb{L}^n \cdot \sum_{r=1}^{n} [\mathcal{X}(n)_r] \cdot \mathbb{L}^r. \tag{8.3.3}$$

Example 8.3.8. If $r = n$ there is only one module, namely $k^{\oplus n}$, where $k = A/\mathfrak{m} = \mathbb{C}$ is the residue field at the origin. Then $[\mathcal{X}(n)_n] = 1/\mathrm{GL}_n$. ◇

Example 8.3.9. The stratum $r = 1$ corresponds to Artinian algebras $A \twoheadrightarrow A/I$, that is, subschemes $Z \subset \mathbb{A}^2$ of length n concentrated at the origin. This gives

$$[\mathcal{X}(n)_1] = \frac{[\mathrm{Hilb}^n(\mathbb{A}^2)_0]}{\mathbb{L}^{n-1}(\mathbb{L}-1)},$$

where we are using that $\mathrm{Aut}_A(\mathscr{O}_Z)$ is an extension of $n-1$ copies of $\mathbb{G}_a$ together with a copy of $\mathbb{G}_m$. This follows easily from [17, Prop. 2.2.1], but cf. [53] or directly Remark 8.3.11 below for a slightly more detailed explanation. Furthermore, the motive of the punctual Hilbert scheme can be extracted from (2.2.7), so the stratum corresponding to $r = 1$ is easily determined. ◇

8.3.2 An inductive strategy

Let us now go back to Conjecture 2 in the form (8.2.4). We already know this formula folds for $n = 0$ and $n = 1$, so it makes sense to prove the formula by induction. Then, after the inductive step, the conjecture becomes equivalent to the relation

$$[\mathcal{E}(n+1)] = \mathbb{L}\cdot[\mathcal{E}(n)] + \mathbb{L}^{n+1}\cdot[\mathcal{C}(n+1)] \text{ for all } n \geq 0. \tag{8.3.4}$$

At this point, the natural attempt would be to exploit Lemma 8.3.6 to write down the left hand side, and compare it with the right hand side of (8.3.4), which is determined by the previous steps along with the Feit–Fine formula. We now show it is enough to do this restricting attention to the "punctual" substacks (see Lemma 8.3.10 below), so for the left hand side we will be able to exploit (8.3.3).

Recall from Notation 8.3.1 the substacks

$$\mathcal{C}(n)_k \subset \mathcal{C}(n)$$

parametrizing coherent sheaves F such that the origin in $\mathbb{A}^2$ appears with multiplicity $n-k$ in the support of F. Then $\mathcal{C}(n)_n$ consist of sheaves without $\mathfrak{m}$ in their support and

$$\mathcal{C}(n)_k = \mathcal{C}(k)_k \times \mathcal{C}(n-k)_0.$$

Form the fibre squares

$$\begin{tikzcd} \mathcal{E}(n)_k \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}(n) \arrow[d, "\pi_n"] \\ \mathcal{C}(n)_k \arrow[r, hook] & \mathcal{C}(n) \end{tikzcd}$$

for $k = 0, 1, \ldots, n$ and observe that $\mathcal{C}(k)_k$ is contained in the stratum $\mathcal{C}(k,k) \subset \mathcal{C}(k)$ over which π_k is a fibration with fibre $\mathbb{C}^k$ (cf. Lemma 8.3.6). Indeed, if a sheaf $[F] \in \mathcal{C}(k)$ does not have $\mathfrak{m}$ in its support, one has

$$\operatorname{Hom}_A(\mathfrak{m}, F) = \operatorname{Hom}_A(\mathscr{O}, F) = H^0(F) = \mathbb{C}^k.$$

It follows that

$$[\mathcal{E}(k)_k] = \mathbb{L}^k \cdot [\mathcal{C}(k)_k] \in K_0(\mathrm{St}_{\mathbb{C}}).$$

Using this relation, we are able to prove the following.

LEMMA 8.3.10. *If one has*

$$[\mathcal{E}(i+1)_0] = \mathbb{L}\cdot[\mathcal{E}(i)_0] + \mathbb{L}^{i+1}\cdot[\mathcal{C}(i+1)_0], \tag{8.3.5}$$

for $i \leq n$, then (8.3.4) holds. In particular, Conjecture 2 is true if and only if (8.3.5) holds for all i.

PROOF. A direct calculation shows that

$$\begin{aligned}
[\mathcal{E}(n+1)] &= \sum_{k=0}^{n+1}[\mathcal{E}(n+1)_k] \\
&= \sum_{k=0}^{n+1}[\mathcal{E}(k)_k]\cdot[\mathcal{E}(n+1-k)_0] \\
&= \sum_{k=0}^{n+1}\mathbb{L}^k\cdot[\mathcal{C}(k)_k]\cdot\big(\mathbb{L}\cdot[\mathcal{E}(n-k)_0]+\mathbb{L}^{n+1-k}\cdot[\mathcal{C}(n+1-k)_0]\big) \\
&= \mathbb{L}\sum_{k=0}^{n}\mathbb{L}^k\cdot[\mathcal{C}(k)_k]\cdot[\mathcal{E}(n-k)_0]+\mathbb{L}^{n+1}\sum_{k=0}^{n+1}[\mathcal{C}(k)_k]\cdot[\mathcal{C}(n+1-k)_0] \\
&= \mathbb{L}\cdot[\mathcal{E}(n)]+\mathbb{L}^{n+1}\cdot[\mathcal{C}(n+1)].
\end{aligned}$$

This recovers the previous inductive form (8.3.4) of Conjecture 2, which is therefore true if and only if (8.3.5) holds for all i. □

It is now easy to verify the base cases of (8.3.5). We quickly do it one more time because we need explicit formulas in order to treat the cases $i>1$ (the argument is inductive). For $i=0$ the right hand side is

$$\mathbb{L}\cdot[\mathcal{E}(0)_0]+\mathbb{L}\cdot[\mathcal{C}(1)_0]=\mathbb{L}+\mathbb{L}\frac{1}{\mathbb{L}-1}=\frac{\mathbb{L}^2-\mathbb{L}+\mathbb{L}}{\mathbb{L}-1}=\frac{\mathbb{L}^2}{\mathbb{L}-1}.$$

On the other hand, the left hand side is

$$[\mathcal{E}(1)_0]=[\mathrm{Hom}_A(\mathfrak{m},k)]\cdot\frac{1}{\mathbb{L}-1}=\frac{\mathbb{L}^2}{\mathbb{L}-1}.$$

We know

$$[\mathcal{C}(2)_0]=\frac{1}{\mathbb{L}-1}+\frac{\mathbb{L}^2}{\mathrm{GL}_2}=\frac{\mathbb{L}^3+\mathbb{L}^2-\mathbb{L}}{\mathrm{GL}_2}.$$

So if $i=1$ we find

$$\mathbb{L}\cdot[\mathcal{E}(1)_0]+\mathbb{L}^2\cdot[\mathcal{C}(2)_0]=\frac{\mathbb{L}^3}{\mathbb{L}-1}+\frac{\mathbb{L}^5+\mathbb{L}^4-\mathbb{L}^3}{\mathrm{GL}_2}=\frac{\mathbb{L}^6+\mathbb{L}^5-\mathbb{L}^3}{\mathrm{GL}_2}.$$

On the other hand,

$$[\mathcal{E}(2)_0]=\mathbb{L}^4\cdot\frac{1}{\mathrm{GL}_2}+\mathbb{L}^3\frac{\mathbb{L}+1}{\mathbb{L}(\mathbb{L}-1)}=\frac{\mathbb{L}^6+\mathbb{L}^5-\mathbb{L}^3}{\mathrm{GL}_2}, \tag{8.3.6}$$

so (8.3.5) holds for $i=0,1$.

8.3.3 The length 3 case

We use the classification of finite A-modules of length 3 entirely supported at the origin, see the joint work [53]. Let $k\cong\mathbb{C}$ be the residue field at the origin $0\in\mathbb{A}^2$. The upshot is that the only indecomposable module of length 3 that is not a structure sheaf is the k-linear dual

$$(A/\mathfrak{m}^2)^*=\mathrm{Hom}_k(A/\mathfrak{m}^2,k)$$

of the (unique) non-curvilinear structure sheaf, defined by the square of the maximal ideal. A quick computation of the hom spaces $\mathrm{Hom}_A(\mathfrak{m},F)$, or an application of Corollary 8.3.7, completes the following table:

r	$\mathcal{C}(3)_0$	$\mathrm{Aut}_A F$	Motivic contribution	$\mathrm{Hom}_A(\mathfrak{m},-)$
1	$\mathscr{O}_Z$	$\mathbb{G}_a^2 \rtimes \mathbb{G}_m$	$\dfrac{\mathbb{L}(\mathbb{L}+1)+1}{\mathbb{L}^2(\mathbb{L}-1)}$	$\mathbb{C}^4$
2	$(A/\mathfrak{m}^2)^*$	$\mathbb{G}_a^2 \rtimes \mathbb{G}_m$	$\dfrac{1}{\mathbb{L}^2(\mathbb{L}-1)}$	$\mathbb{C}^5$
2	$k \oplus \mathscr{O}_Z$	$\mathbb{G}_a^3 \rtimes \mathbb{G}_m^2$	$\dfrac{\mathbb{L}+1}{\mathbb{L}^3(\mathbb{L}-1)^2}$	$\mathbb{C}^5$
3	$k^{\oplus 3}$	GL_3	$\dfrac{1}{\mathrm{GL}_3}$	$\mathbb{C}^6$

Table 1: All $\mathbb{C}[x,y]$-modules of length 3 supported at $\mathfrak{m}$, along with their automorphism groups. Here r is the minimal number of generators.

Remark 8.3.11. The automorphism group of an A-module F of finite length is

$$\mathrm{Aut}_A(F) = U \rtimes \prod_{i=1}^{c} \mathrm{GL}_{m_i}$$

where U is unipotent and $m_1,\ldots,m_c$ are the multiplicities of the indecomposable summands of F. This is proved for instance in [17, Prop. 2.2.1]. In fact, we have been sloppy in Table 1: by $\mathbb{G}_a^j$ in the column "$\mathrm{Aut}_A F$" we actually mean *some* unipotent group of dimension j. However, we only care about the motivic class of $\mathrm{Aut}_A F$, which has become the "denominator" in the next column of the table. Luckily, any unipotent group U in characteristic zero is an iterated extension of copies of $\mathbb{G}_a$. Moreover, the groups $\mathbb{G}_a$ and GL are special, a semi-direct product of special algebraic groups is special, and the motivic class of a semi-direct product of groups is the product of the classes. In particular $\mathrm{Aut}_A F$ is always special, so its class is invertible in $K_0(\mathrm{St}_{\mathbb{C}})$. ♢

The sum of the classes appearing in the third column of Table 1 is

$$\frac{1}{\mathrm{GL}_3}\big(\mathbb{L}^8+\mathbb{L}^7+\mathbb{L}^6-\mathbb{L}^5-\mathbb{L}^4\big), \tag{8.3.7}$$

which matches (as it should) the motive of $\mathcal{C}(3)_0$, as one can check by using the expansion (2.2.8), p. 19. Let us now check the formula

$$\big[\mathcal{E}(3)_0\big] = \mathbb{L}\cdot\big[\mathcal{E}(2)_0\big] + \mathbb{L}^3\cdot\big[\mathcal{C}(3)_0\big]. \tag{8.3.8}$$

Let us start from the right hand side. We have

$$\begin{aligned}
\mathbb{L}\cdot\big[\mathcal{E}(2)_0\big] &= \mathbb{L}\cdot\frac{\mathbb{L}^6+\mathbb{L}^5-\mathbb{L}^3}{\mathrm{GL}_2} && \text{using (8.3.6)}\\
&= \frac{1}{\mathrm{GL}_3}\big(\mathbb{L}^{12}+\mathbb{L}^{11}-2\mathbb{L}^9-\mathbb{L}^8+\mathbb{L}^6\big)\\
\mathbb{L}^3\cdot\big[\mathcal{C}(3)_0\big] &= \frac{1}{\mathrm{GL}_3}\big(\mathbb{L}^{11}+\mathbb{L}^{10}+\mathbb{L}^9-\mathbb{L}^8-\mathbb{L}^7\big) && \text{by (8.3.7)},
\end{aligned}$$

so the right hand side of (8.3.8) is

$$\frac{1}{\mathrm{GL}_3}(\mathbb{L}^{12}+2\mathbb{L}^{11}+\mathbb{L}^{10}-\mathbb{L}^9-2\mathbb{L}^8-\mathbb{L}^7+\mathbb{L}^6).$$

On the other hand, Table 1 allows one to compute the motives of all the strata $\mathcal{X}(3)_r$. Thus applying (8.3.3), we find

$$[\mathcal{E}(3)_0]=\frac{\mathbb{L}^6}{\mathrm{GL}_3}+\mathbb{L}^5\cdot\left(\frac{\mathbb{L}+1}{\mathbb{L}^3(\mathbb{L}-1)^2}+\frac{1}{\mathbb{L}^2(\mathbb{L}-1)}\right)+\mathbb{L}^4\cdot\left(\frac{\mathbb{L}(\mathbb{L}+1)+1}{\mathbb{L}^2(\mathbb{L}-1)}\right),$$

which is easily seen to agree with the previous displayed expression. Thus (8.3.8) is proved.

8.3.4 The length 4 case

The complete classification of $\mathbb{C}[x,y]$-modules of length 4 can be found in [53]. However, in order to establish the formula

$$[\mathcal{E}(4)_0]=\mathbb{L}\cdot[\mathcal{E}(3)_0]+\mathbb{L}^4\cdot[\mathcal{C}(4)_0] \tag{8.3.9}$$

we can simply look at all strata except one: the Feit–Fine formula allows us to compute the last one as well, which we can then substitute in identity (8.3.3) to confirm (8.3.9). In Table 2 below, we as before abuse notation and write $\mathbb{G}_a^j$ for some unipotent group of dimension j.

r	$\mathcal{C}(4)_0$	$\mathrm{Aut}_A(M)$	Motivic contribution	$\mathrm{Hom}_A(\mathfrak{m},-)$
1	$\mathscr{O}_Z$	$\mathbb{G}_a^3\rtimes\mathbb{G}_m$	$\frac{\mathbb{L}^3+2\mathbb{L}^2+\mathbb{L}+1}{\mathbb{L}^3(\mathbb{L}-1)}$	$\mathbb{C}^5$
3	$k^2\oplus\mathscr{O}_Z$	$\mathbb{G}_a^5\rtimes\mathbb{G}_m\times\mathrm{GL}_2$	$\frac{\mathbb{L}+1}{\mathbb{L}^5(\mathbb{L}-1)\mathrm{GL}_2}$	$\mathbb{C}^7$
3	$k\oplus(A/\mathfrak{m}^2)^*$	$\mathbb{G}_a^5\rtimes\mathbb{G}_m^2$	$\frac{1}{\mathbb{L}^5(\mathbb{L}-1)^2}$	$\mathbb{C}^7$
4	$k^{\oplus 4}$	GL_4	$\frac{1}{\mathrm{GL}_4}$	$\mathbb{C}^8$

Table 2: The $\mathbb{C}[x,y]$-modules of length 4 supported at $\mathfrak{m}$, such that $r\neq 2$ (where r is the minimal number of generators), along with their automorphism groups.

Using Table 2 we can write

$$[\mathcal{E}(4)_0]=\frac{\mathbb{L}^3+2\mathbb{L}^2+\mathbb{L}+1}{\mathbb{L}^3(\mathbb{L}-1)}\cdot\mathbb{L}^5+\frac{\mathbb{L}+1}{\mathbb{L}^5(\mathbb{L}-1)\mathrm{GL}_2}\cdot\mathbb{L}^7+\frac{\mathbb{L}^8}{\mathrm{GL}_4}+[\mathcal{X}(4)_2]\cdot\mathbb{L}^6$$

where the motive of $\mathcal{X}(4)_2$ is computed through the Feit–Fine formula (2.2.8) and the knowledge of the other three strata. Using the class of $\mathcal{E}(3)_0$ computed at the previous step, along with the class of $\mathcal{C}(4)_0$, it is a straightforward verification to show that (8.3.9) holds.

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