Virtual invariants of Quot schemes on 3-folds

Andrea Ricolfi
SISSA Trieste

Gauge theory and virtual invariants, Dublin

15 May 2019

Y : smooth 3-fold over $\mathbb{C}$

F : locally free sheaf of rank r on Y

$$\mathrm{Quot}_Y(F, n) = \{\, F \twoheadrightarrow Q \mid \dim Q = 0,\ \chi(Q) = n \,\}.$$

For instance, if $r = 1$,

$$\begin{aligned}\mathrm{Quot}_Y(\mathscr{O}_Y, n) &= \mathrm{Hilb}^n(Y) \\ &= \{\, Z \subset Y \mid Z \text{ finite of length } n \,\}.\end{aligned}$$

Plan of the talk

Show that the Quot scheme

$$\mathrm{Quot}_Y(F, n)$$

carries:

1. a 0-dimensional virtual fundamental class (under some assumptions), and
2. a virtual motive in the sense of Behrend–Bryan–Szendrői.

A VIRTUAL MOTIVE for a scheme X is a motivic weight

$$[X]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})\left[\mathbb{L}^{-\frac{1}{2}}\right]$$

such that $\chi[X]_{\mathrm{vir}} = \chi_{\mathrm{vir}}(X)$, where

$$\chi_{\mathrm{vir}}(X) = \sum_{m \in \mathbb{Z}} m \cdot \chi(\nu^{-1}(m))$$

is the integral of the Behrend function $\nu \colon X(\mathbb{C}) \to \mathbb{Z}$.

1. Virtual fundamental class

We assume Y is projective.
Idea: Identify a quotient $[F \twoheadrightarrow Q] \in \mathrm{Quot}_Y(F, n)$ with its kernel

$$S = \ker(F \twoheadrightarrow Q) \subset F.$$

By "identify", we mean: between moduli of quotients and moduli of kernels, we seek:

- an *isomorphism on tangent spaces*, and
- an *injection on obstruction spaces*.

Assumption : F is simple and rigid.

Apply $\mathrm{Hom}(S, -)$ to $S \hookrightarrow F \twoheadrightarrow Q$:

$$0 \to \mathrm{Hom}(S,S) \to \mathrm{Hom}(S,F) \xrightarrow{u} \mathrm{Hom}(S,Q)$$
$$\xrightarrow{\partial} \mathrm{Ext}^1(S,S) \to \mathrm{Ext}^1(S,F) \to \mathrm{Ext}^1(S,Q) \to \mathrm{Ext}^2(S,S)$$

Assumption : F is simple and rigid.

Apply $\mathrm{Hom}(S, -)$ to $S \hookrightarrow F \twoheadrightarrow Q$:

$$0 \to \mathrm{Hom}(S,S) \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathrm{Hom}(S,Q)$$
$$\xrightarrow{\partial} \mathrm{Ext}^1(S,S) \to 0 \to \mathrm{Ext}^1(S,Q) \to \mathrm{Ext}^2(S,S)$$

So, for F simple and rigid, we have

$$\begin{aligned} \mathrm{Hom}(S, Q) &\xrightarrow{\sim} \mathrm{Ext}^1(S, S) \\ \mathrm{Ext}^1(S, Q) &\hookrightarrow \mathrm{Ext}^2(S, S) \end{aligned}$$

if $S = \ker(F \twoheadrightarrow Q)$.

We're after a *perfect* obstruction theory. If either

(i) Y is Calabi–Yau, or

(ii) $H^{>0}(\mathcal{O}_Y) = 0 = \mathrm{Ext}^{>1}(F, F)$,

then we get

$$\boxed{\mathrm{Hom}(S, S)_0 = \mathrm{Ext}^3(S, S)_0 = 0.}$$

Virtual dimension zero

In both cases (i) and (ii), there is a canonical isomorphism

$$\mathrm{Ext}^1(S,S) \xrightarrow{\sim} \mathrm{Ext}^2(S,S)^*.$$

So the virtual dimension will eventually be

$$\mathrm{vd} = \mathrm{ext}^1 - \mathrm{ext}^2 = 0.$$

The obstruction theory will be symmetric (only) in the CY3 case.

Construction (classical)

$\mathcal{S}$: universal kernel over $Y \times \mathrm{Quot} \xrightarrow{p} \mathrm{Quot}$. Split the trace

$$\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})_0 \to \mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S}) \xrightarrow{\mathrm{tr}} \mathscr{O}_{Y\times \mathrm{Quot}}.$$

The truncated Atiyah class $A(\mathcal{S}) \in \mathrm{Ext}^1(\mathcal{S}, \mathcal{S} \otimes \mathbb{L}_{Y\times\mathrm{Quot}})$ projects, via Verdier duality, to an element

$$\phi \in \mathrm{Hom}(\underbrace{\mathbf{R}p_*(\mathbf{R}\mathscr{H}om(\mathcal{S},\mathcal{S})_0 \otimes \omega_p)[2]}_{\mathbb{E}}, \mathbb{L}_{\mathrm{Quot}}).$$

This map $\mathbb{E} \xrightarrow{\phi} \mathbb{L}_{\mathrm{Quot}}$ is an obstruction theory by [Huybrechts–Thomas] and is perfect by the vanishings $\mathrm{Ext}^i(\mathcal{S},\mathcal{S})_0 = 0$, $i = 0, 3$. ϕ is symmetric in the CY3 case.

Corollary. There is a 0-dimensional virtual class

$$[\mathrm{Quot}_Y(F,n)]^{\mathrm{vir}} \in A_0(\mathrm{Quot}_Y(F,n)),$$

so we can define

$$\mathsf{DT}_{F,n} = \int_{[\mathrm{Quot}_Y(F,n)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Problem: Compute $\mathsf{DT}_F(q) = \sum_{n \geqslant 0} \mathsf{DT}_{F,n} \cdot q^n$

Answer & Speculation

In the Calabi–Yau case, we have

$$\mathsf{DT}_F(q) \overset{(*)}{=} \sum_n \chi_{vir}(\mathrm{Quot}_Y(F,n))q^n \overset{(\diamond)}{=} \mathsf{M}((-1)^r q)^{r\chi(Y)}$$

where $(*)$ is by [Behrend 2005] and $(\diamond)$ is by [Beentjes-R 2018].

In case (ii) (F exceptional), one may ask if the analogue of the $r=1$ case holds:

$$\mathsf{DT}_F(q) \overset{?}{=} \mathsf{M}((-1)^r q)^{rc_3(T_Y \otimes \omega_Y)}$$

(work in progress)

2. Motivic refinement

Y any smooth 3-fold, F any vector bundle on Y. We construct

$$\big[\mathrm{Quot}_Y(F,n)\big]_{\mathrm{vir}} \in \mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})\big[\mathbb{L}^{-\frac{1}{2}}\big]$$

such that $\chi\big[\mathrm{Quot}_Y(F,n)\big]_{\mathrm{vir}} = \chi_{\mathrm{vir}}(\mathrm{Quot}_Y(F,n))$.

$r = 1:$ Behrend–Bryan–Szendrői did this (2013) on $\mathrm{Hilb}^n Y$.

By [Beentjes-R], $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n) = \{d f_n = 0\}$ is a critical locus.
Via motivic vanishing cycles [Denef–Loeser], we get

$$\left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n)\right]_{\mathrm{vir}} = -\mathbb{L}^{-(2n^2+rn)/2}\left[\phi_{f_n}\right]$$

for the "local case", and one can compute (following [BBS])

$$\sum_n \left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n)\right]_{\mathrm{vir}} t^n = \prod_{m=1}^{\infty} \prod_{k=0}^{rm-1} (1 - \mathbb{L}^{2+k-rm/2} t^m)^{-1}.$$

The *relative theory* endows the punctual Quot scheme $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n)_0 \subset \mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n)$ with a motivic weight

$$\mathsf{P}_{r,n} \in \mathcal{M}_{\mathbb{C}}.$$

Define classes $\Omega_{r,n} \in \mathcal{M}_{\mathbb{C}}$ via

$$\mathrm{Exp}\left(\sum \Omega_{r,n} \cdot t^n\right) = \sum (-1)^{rn} \mathsf{P}_{r,n} \cdot t^n.$$

($\star$) Methods developed in [Davison-R, to appear] show

$$\sum_{n\geqslant 0}(-1)^{rn}\left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r,n)\xrightarrow{\text{Hilb - Chow}}\mathrm{Sym}^n\,\mathbb{A}^3\right]_{\mathrm{vir}}$$

$$=\mathrm{Exp}_{\cup}\left(\sum_{n>0}\Omega_{r,n}\boxtimes\left[\mathbb{A}^3\xrightarrow{\Delta_n}\mathrm{Sym}^n\,\mathbb{A}^3\right]\right).$$

($\star\star$) The class

$$\Omega_{r,n}=(-1)^{rn}\mathbb{L}^{-\frac{3}{2}}\cdot\mathbb{L}^{\frac{r(1-n)}{2}}\frac{\mathbb{L}^{rn}-1}{\mathbb{L}^r-1}\left[\mathbb{P}^{r-1}\right]_{\mathrm{vir}}\in\mathcal{M}_{\mathbb{C}}$$

is an effective motive.

... $(\star)$ + $(\star\star)$ imply the power structure identity

$$\sum_n \left[\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n)\right]_{\mathrm{vir}} t^n = \left(\sum_n \mathsf{P}_{r,n} t^n\right)^{\mathbb{L}^3} \in \mathcal{M}_{\mathbb{C}}[[t]].$$

$\rightsquigarrow$ Following [BBS], define $\left[\mathrm{Quot}_Y(F, n)\right]_{\mathrm{vir}}$ by

$$\sum_n \left[\mathrm{Quot}_Y(F, n)\right]_{\mathrm{vir}} t^n = \left(\sum_n \mathsf{P}_{r,n} t^n\right)^{[Y]}.$$

General formula

F vector bundle on a 3-fold. Then

$$\sum_n \left[\mathrm{Quot}_Y(F,n)\right]_{vir}((-1)^r t)^n =$$

$$\mathrm{Exp}\left((-1)^r t \left[Y \times \mathbb{P}^{r-1}\right]_{vir} \mathrm{Exp}\left((-\mathbb{L}^{-\frac{1}{2}})^r t + (-\mathbb{L}^{\frac{1}{2}})^r t\right)\right).$$

Higher rank (motivic) DT invariants (examples)

F : stable rigid vector bundle on a CY3 Y.

Then $\mathrm{Quot}_Y(F,n) \subset M_H^{st}(\mathrm{ch}\, F - (0,0,0,n))$ is a connected component and the rank r classical DT invariants

$$\mathsf{DT}_F(q) = \mathsf{M}((-1)^r q)^{r\chi(Y)}$$

get refined (up to a sign) by the previous formula.

Higher rank Cohomological DT

Say $\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n) = \{ d f_n = 0 \}$. The MHS on the total vanishing cycle cohomology

$$H_c\left(\mathrm{Quot}_{\mathbb{A}^3}(\mathscr{O}^r, n), \Phi_{f_n}\right)$$

is *pure* of Tate type for all n.

Proof. Critical locus description via quivers + a theorem of B. Davison.

Open problem

F : stable rigid vector bundle on a CY3 Y.

Then $\mathrm{Quot}_Y(F, n)$ inherits an oriented d-critical structure [Joyce]. Does there exist an orientation such that the induced virtual motive [Bussi–Joyce–Meinhardt] equals $\left[\mathrm{Quot}_Y(F, n)\right]_{\mathrm{vir}}$?