Higher rank K-theoretic DT theory via Quot schemes

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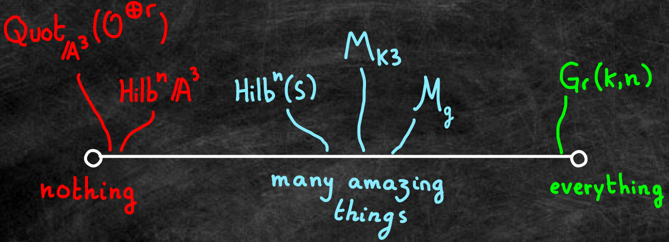
WHAT WE KNOW ABOUT SOME MODULI SPACES



In Donaldson-Thomas theory, moduli spaces we totally don't understand reveal amazing properties.

"Modern enumerative geometry is not so much about numbers as it is about deeper properties of the moduli spaces that parametrize the geometric objects being enumerated."

Andrei Okounkov

Lectures on K-theoretic computations in enumerative geometry

Think beyond numbers

$$\begin{array}{ccc} H^*_c(M, \mathbb{Q}) & & \text{vector space (Hodge structure)} \\ \big\downarrow e_{top} & & \big\downarrow \# \\ M \rightsquigarrow \sum_i (-1)^i b_i(M) & \in & \mathbb{Z} \end{array}$$

DT theory has several natural refinements: we will see the K-theoretic refinement.

DT THEORY: CLASSICAL CONTEXT

$\Lambda^3 \Omega^1_X \cong \mathcal{O}_X$

X : smooth projective Calabi–Yau 3-fold

$\gamma \in H^*(X) \rightsquigarrow M_X(\gamma) =$ moduli space of sheaves with ch $= \gamma$.

$\rightsquigarrow \mathrm{DT}(X,\gamma) \in \mathbb{Z}$ DT invariant

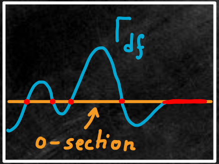
$/\!/$

deformation invariant analogue of $e_{top}(M_X(\gamma))$.

KEY FACT ON $M = M_X(\gamma)$

$$\boxed{M \overset{\text{locally}}{\cong} \{df = 0\} \subseteq \mathcal{U}, \text{ for } \mathcal{U} \text{ a smooth scheme, } f : \mathcal{U} \longrightarrow \mathbb{A}^1.}$$

$\Omega^1_{\mathcal{U}}$



"CRITICAL LOCUS"

$\{df = 0\}$ is virtually 0-dimensional

$\rightsquigarrow$ There is hope to count!

WHAT IS SPECIAL ABOUT $Z = \{df = 0\}$?

(1) $\mathbb{H}^*_c(Z, \Phi_f)$ perverse sheaf of vanishing cycles

$$\mathbb{H}^*_c(Z, \Phi_f) \xrightarrow{e} e_{vir}(Z) = \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbb{H}^i_c(Z, \Phi_f) \in \mathbb{Z}$$

computes DT invariant when $Z = M_X(\gamma)$.

(2) $Z = \{df = 0\}$ has a symmetric obstruction theory:

$$\begin{tikzcd} \mathbb{E}_{crit} \arrow[d, "\varphi"] & = & {[}T_U|_Z \arrow[r, "\mathrm{Hess}(f)"] \arrow[d, "(df)^\vee|_Z"] & \Omega^1_U|_Z{]} \arrow[d, equal] \\ \mathbb{L}_Z & = & {[}\mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] & \Omega^1_U|_Z{]} \end{tikzcd}$$

TRUNCATED COTANGENT COMPLEX

$Z \hookrightarrow U$, $\mathcal{I} \subset \mathcal{O}_U$

$\mathcal{O}_U \xrightarrow{df} \Omega^1_U$

$T_U \xrightarrow{(df)^\vee} \mathcal{I} \subset \mathcal{O}_U$

The obstruction theory φ induces :

(i) a virtual fundamental class $[Z]^{vir} \in A_0(Z)$,

(ii) a virtual structure sheaf $\mathcal{O}_Z^{vir} \in K_0(Z)$.

Remark: the "virtual canonical bundle"

$$K_{vir} := \det \mathbb{E}_{crit} = K_U|_Z^{\otimes 2}$$

has a canonical square root.

ACTION STARTS NOW

Main player in HIGHER RANK DT THEORY OF POINTS is the Quot scheme

$$[S] \in \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$$

$\mathbb{A}^3$ the simplest CY3 ...

$$0 \to S \to \mathcal{O}^{\oplus r} \to T \to 0$$

T: 0-dim sheaf, $\chi(T) = n$

Some facts

(1) $r = 1 \rightsquigarrow$ get $\mathrm{Hilb}^n \mathbb{A}^3$ (local model for 0-dim DT theory)

(2) $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \overset{\text{globally}}{=} \{df = 0\} \subseteq$ explicit $\mathcal{U}_{r,n}$

[Beentjes-R 2018, Szendrői $r=1$]

$\Downarrow$

(3) Have $[\cdot]^{vir}$, $\mathcal{O}^{vir}$, $K_{vir}^{\frac{1}{2}}$

Most important property: Torus action

$$\mathbb{T} = (\mathbb{C}^\times)^3 \times (\mathbb{C}^\times)^r$$

"framing torus"

rescales $\mathcal{O}^{\oplus r}$

moves the support of $T \twoheadleftarrow \mathcal{O}^{\oplus r}$

via $(t_1, t_2, t_3) \cdot (x_1, x_2, x_3) = (t_1 x_1, t_2 x_2, t_3 x_3)$

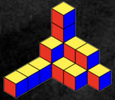
$\Rightarrow$ Get a $\mathbb{T}$-action on $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) =: Q_{r,n}$

The $\mathbb{T}$-fixed locus

$$\left[\bigoplus_{i=1}^{r} I_{\pi_i} \right]$$

$$\rotatebox{90}{$\in$}$$

$$Q_{r,n}^{\mathbb{T}} = \coprod_{n_1+\cdots+n_r=n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathbb{A}^3)^{(\mathbb{C}^\times)^3}$$



plane partitions π_i of size n_i

monomial ideals I_{π_i} of colength n_i

$$\Rightarrow \quad Q_{r,n}^{\mathbb{T}} = \left\{ \begin{array}{c} r\text{-colored plane partitions} \\ \bar{\pi} = (\pi_1, \ldots, \pi_r),\ |\bar{\pi}| = \sum |\pi_i| = n \end{array} \right\}$$

reduced, isolated.

K-THEORETIC INVARIANTS

$\mathbb{E} \to \mathbb{L}_Y$ p.o.t. on a $\mathbb{T}$-scheme $Y \rightsquigarrow \mathcal{O}_Y^{vir} \in K_0^{\mathbb{T}}(Y)$

Y proper $\rightsquigarrow$ $\chi(Y,-) : K_0^{\mathbb{T}}(Y) \to K_0^{\mathbb{T}}(pt)$.

$$\boxed{\chi^{vir}(Y,V) := \chi(Y, V \otimes \mathcal{O}_Y^{vir})}$$

$T_Y^{vir} = \mathbb{E}^\vee$ VIRTUAL TANGENT SPACE,

$N^{vir} = N^{vir}_{Y^{\mathbb{T}}/Y} = T_Y^{vir}\Big|_{Y^{\mathbb{T}}}^{\text{moving}}$ VIRTUAL NORMAL BUNDLE.

VIRTUAL LOCALISATION

Y has a $\mathbb{T}$-action $\Longrightarrow$ $Y^{\mathbb{T}}$ has its own $\mathcal{O}^{vir}$

Fantechi-Göttsche

$$\boxed{\chi^{vir}(Y,V) = \chi^{vir}\left(Y^{\mathbb{T}}, \frac{V|_{Y^{\mathbb{T}}}}{\Lambda^{\bullet} N^{vir,\vee}}\right) \in K_0^{\mathbb{T}}(pt)\left[(1-t^{\mu})^{-1} \,\middle|\, \mu \in \widehat{\mathbb{T}}\right]}$$

If Y is not proper, DEFINE $\chi^{vir}(Y,V)$ to be the RHS, provided that $Y^{\mathbb{T}}$ is PROPER.

Definition of K-theoretic DT invariants of $\mathbb{A}^3$

- $\widehat{\mathcal{O}}^{vir} := \mathcal{O}^{vir} \otimes K_{vir}^{\frac{1}{2}}$ on each $Q_{r,n}$.
- $\mathrm{tr} : K_0^{\mathbb{T}}(pt) \xrightarrow{\sim} \mathbb{Z}[t^{\mu} \mid \mu \in \widehat{\mathbb{T}}]$, $V \mapsto \text{character}(V)$.

virtual localisation

$$DT_{r,n} := \chi(Q_{r,n}, \widehat{\mathcal{O}}^{vir}) = \sum_{[S] \in Q_{r,n}^{\mathbb{T}}} \mathrm{tr}\left(\frac{K_{vir}^{\frac{1}{2}}}{\Lambda^{\bullet}\left((T_S^{vir})^{\vee}\right)} \right)$$

Higher rank K-theoretic DT partition function

$$\kappa := t_1 t_2 t_3$$

$$\mathrm{DT}_r(q) = \sum_{n \geq 0} \mathrm{DT}_{r,n} \cdot q^n \in \mathbb{Z}((t, w, \kappa^{\frac{1}{2}}))[\![q]\!]$$

t_1, t_2, t_3

$w_1, \dots, w_r$ framing parameters

RANK 1: Okounkov proved Nekrasov's Conjecture:

$$DT_1(-q) = \mathrm{Exp}\left(\frac{1}{[\kappa^{\frac{1}{2}} q][\kappa^{\frac{1}{2}} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]} \right)$$

in $\mathbb{Z}((t_1, t_2, t_3, w_1, \kappa^{\frac{1}{2}}))[[q]]$

- Exp = plethystic exponential $\left(\mathrm{Exp}\, f(z_1, \dots, z_s) = \exp \sum_{n \geq 1} \frac{1}{n} f(z_1^n, \dots, z_s^n).\right)$
- $[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.
- Note the independence on w_1 !

THEOREM (FASOLA-MONAVARI-R)

$$\mathrm{DT}_r\left(t, w, \kappa^{\frac{1}{2}}, (-1)^r q\right) = \mathrm{Exp}\ \mathsf{F}_r(t,q), \text{ where}$$

$$\mathsf{F}_r(t,q) := \frac{[\kappa^r]}{[\kappa][\kappa^{\frac{r}{2}} q][\kappa^{\frac{r}{2}} q^{-1}]} \frac{[t_1t_2][t_1t_3][t_2t_3]}{[t_1][t_2][t_3]}.$$

- This was conjectured by Awata-Kanno (2009) in string theory.
- Again: independence on $w_1, \ldots, w_r$.

Before we go on

- A proof of the Awata-Kanno conjecture was also announced (private communication) by Noah Arbesfeld & Yasha Kononov.

- A "10 years later" review on this conjecture was recently arxived by Kanno.

- One more remark on the Physics literature ...

... one can also write

fun fact: same factorisation & shift arise in MOTIVIC DT THEORY

$$DT_r((-1)^r q) = \prod_{i=1}^{r} DT_1\left(-q\kappa^{\frac{-r-1}{2}+i}\right).$$

This formula appeared in work of Nekrasov–Piazzalunga as a limit of (conjectural) DT_4 identities.

Ingredients in the proof

(1) explicit formula $T_S^{vir} = \sum_{i,j=1}^{r} V_{ij} \in K_0^{\Pi}(pt)$, $[S] \in Q^{\Pi}_{r,n}$.

! (2) $DT_r(t, w, \kappa^{\frac{1}{2}}, q)$ does not depend on w.

(3) Evaluate $DT_r(t, w, \kappa^{\frac{1}{2}}, q) = \sum_{\bar{\pi}} q^{|\bar{\pi}|} \prod_{i,j=1}^{r} [-V_{ij}]$.

by setting $w_i = L^i$ and letting $L \longrightarrow \infty$.

Higher rank vertex

T_S^{vir} is a sum of $V_{ij} = w_i^{-1} w_j \left(Q_j - \frac{\overline{Q}_i}{\kappa} + \frac{(1-t_1)(1-t_2)(1-t_3)}{\kappa} Q_j \overline{Q}_i \right)$

where $Q_i = \mathrm{tr}_{\mathcal{O}/I_{\pi_i}} = \sum_{\square \in \pi_i} t^{\square}$, and $\overline{(\cdot)}$ sends $t_i \to t_i^{-1}$.

To run this higher rank version of [MNOP], we check that $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbb{T}}$ carries the TRIVIAL OBSTRUCTION THEORY.

Cohomological DT invariants

$s_i = c_1^{\mathbb{T}}(t_i)$, $v_j = c_1^{\mathbb{T}}(w_j)$

$$\mathrm{DT}^{\mathrm{coh}}_{r,n} := \int_{[Q_{r,n}]^{\mathrm{vir}}} 1 := \sum_{[S]\in Q^{\mathbb{T}}_{r,n}} e^{\mathbb{T}}(-T_S^{\mathrm{vir}}) \in \mathbb{Q}((s,v))$$

Szabo's Conjecture

[MNOP]: $r=1$ ✓
[Benini-Bonelli-Poggi-Tanzini]: verified up to $r,n \leq 8$

$$\boxed{\mathrm{DT}^{\mathrm{coh}}_{r}(q) := \sum_{n\geq 0} \mathrm{DT}^{\mathrm{coh}}_{r,n}\cdot q^n = M((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1s_2s_3}}}$$

$\left(M(q) := \sum_{\pi} q^{|\pi|} = \prod_{m\geq 1}(1-q^m)^{-m} \quad \text{MacMahon function} \right)$

THEOREM (FASOLA-MONAVARI-R). SZABO'S CONJECTURE IS TRUE.

"Proof"

(1) $\mathrm{DT}_r^{\mathrm{coh}}(q) = \lim_{b \to 0} \mathrm{DT}_r\left(e^{bs}, e^{bv}, q\right)$,

(2) $\mathrm{DT}_r^{\mathrm{coh}}$ does not depend on $v = e^{\mathbb{T}}(w)$,

(3) compute $\lim_{b \to 0} \mathrm{Exp}\, F_r(t_1^b, t_2^b, t_3^b, q)$. $\square$

FUTURE (?) OF DT THEORY OF POINTS

?

$\downarrow$ $p=0$

✓ K-THEORETIC

$\downarrow$ e^{T}

✓ COHOMOLOGICAL

$\downarrow$ $s_1+s_2+s_3=0$

ENUMERATIVE

We propose a definition of VIRTUAL CHIRAL ELLIPTIC GENUS
(math. formulation of ELLIPTIC DT INVARIANTS [Benini-Bonelli-Poggi-Tanzini]

... Guess a formula!

One more formula awaits proof:

$$\sum_{n\geq 0} q^n \int_{[\mathrm{Quot}_Y(F,n)]^{\mathrm{vir}}} 1 = M((-1)^r q)^{r\cdot c_3(T_Y\otimes\omega_Y)}$$

(Y projective 3-fold, F exceptional)

Final observation (with Alberto Cazzaniga)

Fix a hyperplane $D \subset \mathbb{P}^m$, $m \geq 3$. Let $\mathrm{Fr}_{r,n}(\mathbb{P}^m)$ be the moduli space of D-framed sheaves on $\mathbb{P}^m$, i.e. pairs (E, φ) where $\mathrm{ch}(E) = (r, 0, \ldots, 0, -n)$ and $\varphi : E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus r}$.

There is an isomorphism

$$\eta : \mathrm{Quot}_{\mathbb{A}^m}(\mathcal{O}^{\oplus r}, n) \xrightarrow{\sim} \mathrm{Fr}_{r,n}(\mathbb{P}^m)$$

$$[E \overset{i}{\hookrightarrow} \mathcal{O}_{\mathbb{P}^m}^{\oplus r} \twoheadrightarrow T] \longmapsto (E, i|_D)$$

Proof

(1) η is bijective. Easy ($E \hookrightarrow E^{\vee\vee} = \mathcal{O}_{\mathbb{P}^m}^{\oplus r}$), and false if $m=2$.

(2) at $[E] \overset{\eta}{\mapsto} (E, i|_D)$: tangents: $\mathrm{Hom}(E,Q) \xrightarrow[\sim]{d\eta} \mathrm{Ext}^1(E, E(-D))$

obstructions: $\mathrm{Ext}^1(E,Q) \hookrightarrow \mathrm{Ext}^2(E,E(-D))$

$\Downarrow$

(3) Get isomorphism of LOCAL DEFORMATION FUNCTORS.

$\Downarrow$

(4) η is ÉTALE. $\square$

Thank you !!