Moduli of semiorthogonal decompositions

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Overview

§ 0. Derived / triangulated categories

§ 1. Semiorthogonal decompositions (SOD)

§ 2. Main result: there is a moduli space of SODs.

§ 3. First applications

§ 0. DERIVED CATEGORIES

$\mathcal{A}$: abelian category such as $\mathrm{Coh}(Y)$, $\mathrm{QCoh}(Y)$ for Y a noetherian scheme

$D(\mathcal{A})$: complexes $A^\bullet = (\cdots \to A^i \to A^{i+1} \to \cdots)$ with somewhat exotic morphisms ...

$\uparrow$ loc (sends quasi-isomorphisms to ISOMORPHISMS)

$K(\mathcal{A})$: homotopy category of $\mathcal{A}$ (Homs = chain maps / homotopy)

$$\mathcal{A} \xhookrightarrow{\text{FULL}} D(\mathcal{A})$$

$$E \mapsto \big(\cdots \to \underset{-1}{0} \to \underset{0}{E} \to \underset{1}{0} \to \cdots\big)$$

TWO VARIATIONS ON $D(Coh(Y))$

EQUIVALENCE for SMOOTH VARIETIES

$$\mathrm{Perf}(Y) \longrightarrow D^b(Y) \subset D(Coh(Y))$$

$\mathrm{Perf}(Y) = \{$perfect complexes$\}$

$D^b(Y) = \{$bounded complexes$\}$

perfect: locally quasi-isomorphic to a bounded complex of locally free sheaves

Motivation for $D^b(Y)$

General principle: a variety is determined by sheaves on it.

$$Y \cong Y' \iff \mathrm{Coh}(Y) \cong \mathrm{Coh}(Y')$$

Look at half-empty glass: $\mathrm{Coh}(-)$ is a coarse invariant!

But $D^b(-)$ is a finer invariant:

Y smooth projective, $\pm K_Y$ ample.
Then (Bondal-Orlov) $D^b(Y) \cong D^b(Y') \Rightarrow Y \cong Y'$.

... All these categories $\mathcal{T} \in \{ D^b(Y), D(\mathrm{Coh}(Y)), \mathrm{Perf}(Y), D(\mathcal{A}) \}$ are TRIANGULATED, i.e. equipped with an autoequivalence $[1] : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ and a class of EXACT TRIANGLES $E^\bullet \to F^\bullet \to G^\bullet \to E^\bullet[1]$ satisfying some axioms.

"upgrade" of SHORT EXACT SEQUENCES

$$\boxed{A, B \in \mathcal{A} \implies \mathrm{Ext}^i_{\mathcal{A}}(A,B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])}$$

$f : E^\bullet \to F^\bullet$ map of complexes in $\mathcal{A}$

$\rightsquigarrow$ $\mathrm{cone}(f)^i = E^{i+1} \oplus F^i$, $\quad \mathrm{cone}(f)^i \xrightarrow{\begin{pmatrix} -d_{E^\bullet}^{i+1} & 0 \\ f^{i+1} & d_{F^\bullet}^i \end{pmatrix}} \mathrm{cone}(f)^{i+1}$

$\rightsquigarrow$ $\mathrm{cone}(f)$ new complex. ALL (!) EXACT TRIANGLES in $D(\mathcal{A})$ are (up to iso) of the form

$$E^\bullet \xrightarrow{f} F^\bullet \longrightarrow \mathrm{cone}(f) \longrightarrow E^\bullet[1]$$

$$F^i \to E^{i+1} \oplus F^i \qquad\qquad E^{i+1} \oplus F^i \to E^{i+1}$$

CONES

e.g. $\quad 0 \to B \to E \to A \to 0$ in $\mathcal{A}$ gives an extension ε.

$\rightsquigarrow$ $B \to E \to A \xrightarrow{p} B[1]$ EXACT TRIANGLE in $D(\mathcal{A})$

$$\varepsilon \in \mathrm{Ext}^1(A,B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[1])$$

Also, derived functors

Idea: π_*, π^*, $\otimes$, $\mathcal{H}om$ not exact $\rightsquigarrow$ replace them with functors sending exact triangle $\mapsto$ exact triangle

- $R\pi_* : D(QCoh_X) \to D(QCoh_Y)$
- $L\pi^* : D(QCoh_Y) \to D(QCoh_X)$
- $D(QCoh_X) \times D(QCoh_X) \xrightarrow{- \overset{L}{\otimes} -} D(QCoh_X)$
- $D(QCoh_X) \times D(QCoh_X) \xrightarrow{R\mathcal{H}om(-,-)} D(QCoh_X)$

$X \xrightarrow{\pi} Y$ morphism of schemes (+ assumptions)

L: left derived

R: right derived

§ 1 Semiorthogonal decompositions

$\mathcal{T}$: fixed triangulated category

$\mathcal{T}_1, \dots, \mathcal{T}_n \subset \mathcal{T}$ full subcategories $\rightsquigarrow$

$\boxed{\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle \text{ is a SOD of } \mathcal{T}}$

DEF.

- $\mathcal{T}_1, \dots, \mathcal{T}_n$ generate $\mathcal{T}$
- $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_i, \mathcal{T}_j) = 0, \quad i > j$

no homs !

"generate": $\mathcal{T}$ is equivalent (via inclusion) to the smallest triangulated subcategory containing $\mathcal{T}_1, \dots, \mathcal{T}_n$.

WHAT DOES IT MEAN ?

It means that $\mathcal{T}_1, \ldots, \mathcal{T}_n$ can be used to "decompose" any $T \in \mathcal{T}$ in the following sense: $\exists!$ "FILTRATION"

$$\boxed{\begin{gathered} 0 = T_n \to T_{n-1} \to \cdots \to T_0 = T \\ \text{with } \operatorname{cone}(T_i \to T_{i-1}) \in \mathcal{T}_i \end{gathered}}$$

$\rightsquigarrow$ PROJECTION FUNCTORS (of the given SOD)

$$p_{\mathcal{T}_i} : \mathcal{T} \to \mathcal{T}_i \hookrightarrow \mathcal{T}$$
$$T \mapsto \operatorname{cone}(T_i \to T_{i-1})$$

Example

$\boxed{\mathcal{T} = \langle \mathcal{T}, 0 \rangle \text{ and } \mathcal{T} = \langle 0, \mathcal{T} \rangle}$ are called the TRIVIAL SODs.

Example

$\mathcal{S} \xhookrightarrow{i} \mathcal{T}$ admissible subcategory (i.e. have left, right adjoints $\mathcal{T} \underset{i^!}{\overset{i^*}{\rightrightarrows}} \mathcal{S}$)

$${}^{\perp}\mathcal{S} = \{ T \in \mathcal{T} \mid \mathrm{Hom}(T, A[e]) = 0 \quad \forall A \in \mathcal{S}, \ \forall e \in \mathbb{Z} \}$$

$$\mathcal{S}^{\perp} = \{ T \in \mathcal{T} \mid \mathrm{Hom}(A[e], T) = 0 \quad \forall A \in \mathcal{S}, \ \forall e \in \mathbb{Z} \}$$

$\rightsquigarrow$ two SODs $\boxed{\mathcal{T} = \langle \mathcal{S}, {}^{\perp}\mathcal{S} \rangle, \quad \mathcal{T} = \langle \mathcal{S}^{\perp}, \mathcal{S} \rangle .}$

Example

$Y = \mathbb{P}^n, \quad \mathcal{T} = D^b(\mathbb{P}^n), \quad \mathcal{T}_j = \langle \mathcal{O}(j) \rangle \quad (0 \leq j \leq n)$

Beilinson : $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle.$

"full exceptional sequence"
(stronger than SOD)

or: $\langle \mathcal{O}(i), \mathcal{O}(i+1), \dots, \mathcal{O}(i+n) \rangle, \quad i \in \mathbb{Z}$

- $E^\bullet \in D^b(\mathbb{P}^n)$ exceptional: $\mathrm{Hom}(E^\bullet, E^\bullet[e]) = \begin{cases} \mathbb{C} & e = 0 \\ 0 & e \neq 0 \end{cases}$
- $E_1^\bullet, \dots, E_n^\bullet$ exceptional sequence: $E_i^\bullet$ exceptional, $\mathrm{Hom}(E_i^\bullet, E_j^\bullet[e]) = 0, \ i > j, \ \forall e.$

  no homs !

- FULL ($\leftrightarrow$ GENERATION): consequence of Beilinson spectral sequences

$$\left[\begin{array}{ll} E_1^{r,s} := H^s(F(r)) \otimes \Omega^{-r}(-r) \Longrightarrow E^{r+s} = \begin{cases} F & r+s=0 \\ 0 & \text{else} \end{cases} & \\ E_1^{r,s} := H^s(F \otimes \Omega^{-r}(-r)) \otimes \mathcal{O}(r) \Longrightarrow E^{r+s} = \begin{cases} F & r+s=0 \\ 0 & \text{else} \end{cases} & F \in \mathrm{Coh}(\mathbb{P}^n) \end{array}\right]$$

Example

X smooth variety, $Y \underset{\text{codim } c}{\overset{lci}{\hookrightarrow}} X$, take blowup

$$\begin{tikzcd} D \arrow[r, hook, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & \tilde{X} \arrow[d, "\pi"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

$\mathcal{O}_D(1) \to D \cong \mathbb{P}(\mathcal{N}_{Y/X})$

$\otimes k$

fully faithful $\forall k \in \mathbb{Z}$

$L\pi^*: D^b(X) \to D^b(\tilde{X})$

$\psi_k : D^b(Y) \to D^b(\tilde{X})$, $F \mapsto Ri_*(Lp^*(F) \overset{L}{\otimes} \mathcal{O}_D(k))$

$$\rightsquigarrow \quad \boxed{D^b(\tilde{X}) = \langle \underset{\mathcal{T}_1}{\underset{\parallel}{L\pi^* D^b(X)}}, \underset{\mathcal{T}_2}{\underset{\parallel}{\psi_0(D^b(Y))}}, \dots, \underset{\mathcal{T}_c}{\underset{\parallel}{\psi_{c-2}(D^b(Y))}} \rangle}$$

Indecomposability

One always has $\mathcal{T} = \langle \mathcal{T}, 0 \rangle$ and $\mathcal{T} = \langle 0, \mathcal{T} \rangle$.
If these are the only SODs, $\mathcal{T}$ is called INDECOMPOSABLE.

$\mathcal{T} = D^b(Y)$ indecomposable for:

(1) Y = curve of genus $\geqslant 1$ (Okawa)

(2) Y smooth connected, $K_Y = 0$ (Bridgeland)

(3) Y smooth, proper, $Bs\,|\omega_Y|$ finite (Kawatani-Okawa)

(4) ... More in §3

§ 2. Moduli of SODs

How do SODs behave in smooth families?

i.e. given $D^b(X) = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle$ and a family
what can we say about SODs of X_η?

A consequence of our main result is: IF $\mathcal{U}$ IS IRREDUCIBLE AND THE GENERIC FIBRE IS INDECOMPOSABLE, THEN SO IS THE SPECIAL FIBRE.
[Bastianelli - Belmans - Okawa - R]

excellent scheme
e.g. $\mathbb{C}$-variety

Input: a smooth projective morphism $\mathcal{X} \xrightarrow{f} \mathcal{U}$, $n \geq 2$

Output: a functor $\mathrm{SOD}_f^n : \mathrm{Sch}_{\mathcal{U}}^{op} \longrightarrow \mathrm{Sets}$

$$\mathrm{SOD}_f^n(V \to \mathcal{U}) = \left\{ V\text{-linear SODs } \mathrm{Perf}(\mathcal{X} \times_{\mathcal{U}} V) = \langle \mathcal{T}_1, \dots, \mathcal{T}_n \rangle \right\}$$

this is the def. for $(V \to \mathcal{U}) \in \mathrm{Aff}_{\mathcal{U}}$

V-linear means: the components $\mathcal{T}_i$ are V-linear i.e.

$$Lf_V^*(\mathrm{Perf}(V)) \overset{L}{\otimes} \mathcal{T}_i \subseteq \mathcal{T}_i$$

$$\begin{tikzcd}
\mathcal{X}_V = \mathcal{X} \times_{\mathcal{U}} V \arrow[r] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
V \arrow[r] & \mathcal{U}
\end{tikzcd}$$

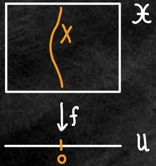
Linearity is needed to make sure we can base-change SODs [Kuznetsov]

THEOREM (BOR) SOD^n_f is an algebraic space, étale over $\mathcal{U}$.

[RELATED: Antieau-Elmanto construct a STACK of SODs in a more general setup.]

FROM NOW ON: $n = 2$

Corollary: ∃! deformation of given SOD



Given $D^b(X) \overset{(*)}{=} \langle \mathcal{A}, \mathcal{B} \rangle$, possibly after passing to an étale neighborhood $\mathcal{U}' \to \mathcal{U}$ of $o \in \mathcal{U}$,

$\exists!$ $\mathcal{U}$-linear SOD $\mathrm{Perf}\, \mathfrak{X} = \langle \mathcal{A}_u, \mathcal{B}_u \rangle$

whose base change along $\mathrm{Spec}\, k(o) \to \mathcal{U}$ is $(*)$.

[RELATED: Hu proved this for exceptional sequences.]

Example $\mathcal{X} \xrightarrow{f} \mathcal{U}$ family of Calabi-Yau varieties

$\Rightarrow \quad \mathrm{SOD}^2_f = \mathcal{U} \amalg \mathcal{U} \to \mathcal{U}.$

Example $\mathcal{X} = \mathbb{P}^1 \xrightarrow{f} \mathcal{U} = pt$

$\langle \mathcal{O}(i), \mathcal{O}(i+1) \rangle$

$$\rightsquigarrow \quad \mathrm{SOD}^2_f = \underbrace{pt \amalg pt}_{\text{trivial SODs}} \amalg \coprod_{i \in \mathbb{Z}} pt$$

not quasi-compact ...

Proof (Artin's criterion)

$P : Sch_U^{op} \to Sets$ presheaf. It is an algebraic space étale over U iff:

(1) P is a sheaf on $(Sch_U)_{ét}$

TWO WORDS ON THIS BELOW...

(2) P is locally of finite presentation (limit preserving)

(3) (B, m) local noetherian ring, m-complete,
$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] & U \\ \operatorname{Spec} B/m \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

$$\Rightarrow P(\operatorname{Spec} B) \xrightarrow{\sim} P(\operatorname{Spec} B/m).$$

TWO WORDS ON LIMIT PRESERVING

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{U}} \mathcal{X} = \mathcal{Y} \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{X} \arrow[r, "f"'] & \mathcal{U}
\end{tikzcd}$$

$K \in \mathrm{Perf}(\mathcal{Y}) \rightsquigarrow$ FOURIER-MUKAI FUNCTOR

$$\phi_K : \mathrm{Perf}(\mathcal{X}) \longrightarrow \mathrm{Perf}(\mathcal{X})$$
$$E \longmapsto Rp_{2*}(p_1^* E \otimes K)$$

$\boxed{\text{Notation: } \mathcal{E}_K \subseteq \mathrm{Perf}(\mathcal{X}) \text{ the essential image of } \phi_K.}$

Consider the functors $\mathrm{DEC}_{\Delta_f} \subseteq \mathbb{F}_{\mathcal{O}_{\Delta_f}} : \mathrm{Aff}_u^{op} \longrightarrow \mathrm{Sets}$ given by

$$\mathrm{DEC}_{\Delta_f}(V \to U) = \left\{ \begin{array}{l} \text{EXACT TRIANGLES } K_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}} \\ \text{SUCH THAT } Rf_{V*} R\mathcal{H}om\left(\mathcal{E}_{K_{\mathcal{B}}}, \mathcal{E}_{K_{\mathcal{A}}}\right) = 0 \end{array} \right\} \Big/ \cong$$

$$\text{in}$$

$$\mathbb{F}_{\mathcal{O}_{\Delta_f}}(V \to U) = \left\{ \text{EXACT TRIANGLES } K_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}} \right\} / \cong$$

$$\begin{tikzcd}
\mathcal{Y}_V \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_V \arrow[r] \arrow[d, "f_V"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{X}_V \arrow[r, "f_V"'] & V \arrow[r] & U
\end{tikzcd}$$

In general, we show that if $Y \to U$ is smooth and $G \in \mathrm{Perf}(Y)$ then the functor $\mathbb{F}_G : (V \to U) \mapsto \{\text{EXACT TRIANGLES } K \to G_V \to L\}/\cong$ is LIMIT PRESERVING. So $\mathbb{F}_{\mathcal{O}_{\Delta_f}}$ is LIMIT PRESERVING.

$$\mathrm{DEC}_{\Delta_f} \cong \mathbb{F}_{\mathcal{O}_{\Delta_f}}$$

UPSHOT: DEC_{Δ_f} IS ALSO LIMIT PRESERVING!

Now, crucially, $\mathrm{DEC}_{\Delta_f} \cong \mathrm{SOD}_f \big|_{\mathrm{Aff}_U^{op}}$ ← JUST 2 WORDS ON THIS ...

("Aff" is enough, since $\mathrm{Sh}(\mathrm{Aff}_U)_{\acute{e}t} \cong \mathrm{Sh}(\mathrm{Sch}_U)_{\acute{e}t} \dots$)

$$\mathrm{DEC}_{\Delta_f} \cong \mathrm{SOD}_f \big|_{\mathrm{Aff}_u^{op}}$$

$\subseteq$ Given $[K_{\mathcal{B}} \to \mathcal{O}_{\Delta_{f_V}} \to K_{\mathcal{A}}] \in \mathrm{DEC}_{\Delta_f}(V \to U)$, we get $\mathrm{Hom}_{\mathcal{X}_V}(\mathcal{E}_{K_{\mathcal{B}}}, \mathcal{E}_{K_{\mathcal{A}}}) = 0$.

$(\mathcal{E}_{K_{\mathcal{A}}}, \mathcal{E}_{K_{\mathcal{B}}}) \in \mathrm{SOD}^2_f(V \to U) \Leftarrow$ they generate $\mathrm{Perf}(\mathcal{X}_V) \Leftarrow$ apply $\Phi_{\mathcal{O}_{\Delta_{f_V}}}$

$\supseteq$ Given $(V \to U) \in \mathrm{Aff}_U$ and $(\mathcal{A}, \mathcal{B}) \in \mathrm{SOD}^2_f(V \longrightarrow U)$

$$\mathcal{B} \underset{i^!}{\overset{i}{\rightleftarrows}} \mathrm{Perf}(\mathcal{X}_V) \underset{j^*}{\overset{j}{\leftrightarrows}} \mathcal{A} \qquad P_{\mathcal{A}} = j \circ j^*, \quad P_{\mathcal{B}} = i \circ i^! \quad \text{PROJECTION FUNCTORS}$$

$$\mathcal{O}_{\Delta_{f_V}} \in \mathrm{Perf}(\mathcal{Y}_V) = \langle p_2^* \mathcal{A}, p_2^* \mathcal{B} \rangle \quad \mathcal{X}_V\text{-linear SOD}$$

$$\rightsquigarrow \quad K_{\mathcal{B}} \longrightarrow \mathcal{O}_{\Delta_{f_V}} \longrightarrow K_{\mathcal{A}} \ \text{ in } \mathrm{Perf}(\mathcal{Y}_V), \ \text{satisfies } Rf_{V*} R\mathcal{H}om = 0$$

Easy check: $(\mathcal{E}_{K_{\mathcal{A}}}, \mathcal{E}_{K_{\mathcal{B}}}) = (\mathcal{A}, \mathcal{B})$ $\square$

§ 3. APPLICATIONS [BOR + F. Bastianelli] $/\mathbb{C}$

$\boxed{\text{If } U \text{ is irreducible, generic fibre of } \mathcal{X} \xrightarrow{f} U \text{ is indecomposable} \implies \text{ALL FIBRES ARE INDECOMPOSABLE}}$

$$\begin{tikzcd} \{\text{nontrivial SODs}\} \arrow[r, hook, "\text{open}"] \arrow[dr, "\text{ét}"'] & \mathrm{SOD}^2_f \arrow[d, "\text{ét}"] \\ & U \end{tikzcd}$$

U irreducible

$\Rightarrow \{u \in U \mid \mathcal{X}_u \text{ has a nontrivial SOD}\} =: T \overset{\text{open}}{\hookrightarrow} U$

If the generic fibre $\mathcal{X}_\eta$ is indecomposable

$\Rightarrow \{u \in U \mid \mathcal{X}_u \text{ INDECOMPOSABLE}\} =: T' \overset{\text{dense}}{\hookrightarrow} U$

$T \cap T' = \emptyset \Rightarrow T = \emptyset$. $\square$

Example

$\mathcal{X} \xrightarrow{f} \mathcal{U}$, $\operatorname{Bs}|\omega_{\mathcal{X}_0}|$ finite for some $0 \in \mathcal{U}$ $\xRightarrow[\text{Kawatani-Okawa}]{\text{using}}$ ALL FIBRES ARE INDECOMPOSABLE

Example

S smooth projective surface. If $\operatorname{Bs}|\omega_S| = \emptyset$ then $\operatorname{Bs}|\omega_{\mathrm{Hilb}^n S}| = \emptyset$

So $\mathrm{Hilb}^n(S)$ is INDECOMPOSABLE $\forall n \geq 1$ as soon as $\operatorname{Bs}|\omega_S| = \emptyset$.

Example

C: smooth projective curve of genus $g \geq 2$

Fix $1 \leq n < \lfloor \frac{g+3}{2} \rfloor$ $\implies$ $D^b(\mathrm{Sym}^n C)$ is INDECOMPOSABLE

(proved also by [Biswas-Gomez-Lee] for $n < \mathrm{gon}(C)$)

We expect indecomposability for $n \leq g-1$.

PROOF $\operatorname{gon}(C) \leq \left\lfloor \frac{g+3}{2} \right\rfloor$ and a general curve realises this bound.

$$\begin{tikzcd} \mathcal{C}_\eta \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\text{smooth}"', "\pi"] & C \arrow[l] \arrow[d] \\ \text{generic pt } \eta \arrow[r] & U & 0 \arrow[l] \end{tikzcd}$$

generic fibre of gonality $= \left\lfloor \frac{g+3}{2} \right\rfloor$

But $n < \operatorname{gon}(C_u) \iff Bs\,|\omega_{\operatorname{Sym}^n C_u}| = \emptyset \quad u \in U$

Now set $1 \leq n < \left\lfloor \frac{g+3}{2} \right\rfloor$

$\rightsquigarrow \mathcal{X} = \operatorname{Sym}^n(\pi) \to U$ smooth family, generic fibre has empty canonical base locus $\Rightarrow$ all fibres are indecomposable $\boxtimes$

Thank you!