# Higher rank motivic Donaldson-Thomas invariants

Università di Trieste, 19/2/2021

Andrea Ricolfi

SISSA

# THE GROTHENDIECK RING OF VARIETIES

$$K_0(\mathrm{Var}) := \frac{\mathbb{Z}\cdot\{\text{iso. classes } [Y] \mid Y \text{ complex variety}\}}{[Y] = [Y\setminus Z] + [Z],\ Z\subset Y \text{ closed}}$$

$[Y] \in K_0(\mathrm{Var})$ : the <u>MOTIVE</u>, or <u>UNIVERSAL EULER CHARACTERISTIC</u> of $Y$

UNIVERSAL PROPERTY OF $K_0(\mathrm{Var})$

$$\begin{tikzcd}
\mathrm{Var} \arrow[r, "Y \mapsto {[Y]}"] \arrow[d, "\forall e"'] & K_0(\mathrm{Var}) \arrow[dl, dashed, "\exists!\ \text{ring map}"] \\
\text{any ring } R &
\end{tikzcd}$$

$e(\emptyset) = 0_R$

$e(pt) = 1_R$

$e(Y\times Y') = e(Y)\cdot e(Y')$

$e(Y) = e(Z) + e(Y\setminus Z)$

"The ring $K_0(\mathrm{Var})$ is interesting, big, and hard to grasp"

Looijenga, Motivic measures

# Computing in $K_0(\mathrm{Var})$

Rules:

(0) It is hard.

(1) $X \to Y$ bijective morphism $\Longrightarrow$ $[X] = [Y]$

(2) $X \to Y$ Zariski locally trivial $\Longrightarrow$ $[X] = [Y] \cdot [\text{fibre}]$

The most important class is the Lefschetz motive $\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var})$

$$[\mathbb{P}_Y(E)] \overset{(2)}{=} [Y] \cdot (1 + \mathbb{L} + \cdots + \mathbb{L}^r)$$

locally free of rank $r+1$

# $K_0(\mathrm{Var})$ "SEES" SOME GEOMETRY

2014 Galkin-Shinder

BUT IT IS...

EXPECTED

If $\mathbb{L}$ is <u>not</u> a 0-divisor $\Rightarrow$ the generic cubic $Y \subset \mathbb{P}^5$ is irrational.

2017 Kuznetsov-Shinder CONJECTURE

if $X, Y$ are smooth proj. simply connected manifolds, then

$$D^b(X) \cong D^b(Y) \Rightarrow \mathbb{L}^a([X]-[Y]) = 0 \in K_0(\mathrm{Var}),\ \text{some } a \geq 0.$$

... EVEN THOUGH IT DOES NOT LOOK SO :

By definition $[X] = [X_{red}]$ so <u>SCHEME</u> theory is invisible

2003 Larsen-Lunts CUT-AND-PASTE CONJECTURE

if $[X] = [Y] \in K_0(\mathrm{Var})$ then one has stratifications

$$X = \coprod_i X_i,\quad Y = \coprod_i Y_i \quad \text{s.t. } X_i \cong Y_i$$

2015 Borisov Disproved this VERY REASONABLE conjecture ...

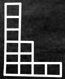
# POWER STRUCTURES, $R = \mathbb{Z}$

Take a power series $A(t) = 1 + \sum_{n>0} A_n t^n \in 1 + t\mathbb{Z}[[t]]$, $m \in \mathbb{N}$

$$A(t)^m = 1 + \sum_{n>0} \sum_{\alpha \vdash n} \left( \prod_{j=0}^{\|\alpha\|-1} (m-j) \cdot \frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i !} \right) t^n$$

where $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots s^{\alpha_s})$ are PARTITIONS. $\|\alpha\| := \sum_i \alpha_i$, $n = \sum_i i\alpha_i$

e.g. $\alpha = (1^3 2^2 4^1) \vdash 11 \quad \longleftrightarrow$



$\|\alpha\|$ = number of rows in the Young diagram $= 3 + 2 + 1$

# Power structures, semiring $R$

A power structure on $R$ is a map
$$(1+tR[[t]]) \times R \longrightarrow 1+tR[[t]]$$
$$(A(t), m) \longmapsto A(t)^m$$

such that:

- $A(t)^0 = 1$
- $A(t)^1 = A(t)$
- $(A(t)\cdot B(t))^m = A(t)^m \cdot B(t)^m$
- $A(t)^{m+m'} = A(t)^m \cdot A(t)^{m'}$
- $(A(t)^m)^{m'} = A(t)^{m\cdot m'}$
- $(1+t)^m = 1+mt \pmod{t^2}$
- $A(t)^m \big|_{t\mapsto t^e} = A(t^e)^m$

# POWER STRUCTURES, $R = K_0(Var)$

Say $\{A_n\}_{n>0}$, $M$ are VARIETIES.

$$\left(1+\sum_{n>0}[A_n]t^n\right)^{[M]} := 1+\sum_{n>0}\sum_{\alpha\vdash n}\left[\frac{\left(\prod_i M^{\alpha_i}\right)\setminus\Delta^{big}\times\prod_i A_i^{\alpha_i}}{\prod_i \mathfrak{S}_{\alpha_i}}\right]t^n$$

Gusein-Zade
Luengo
Melle-Hernández

This defines a POWER STRUCTURE on the sub-semiring $S_0(Var) \subset K_0(Var)$ of effective classes. It extends uniquely to a power structure on $K_0(Var)$ since any $A \in K_0(Var)$ can be written $A = B - C$, $B, C \in S_0(Var)$:

$$A(t)^M = B(t)^M \cdot \left(C(t)^M\right)^{-1} \quad \text{works!}$$

# MOTIVIC (PLETHYSTIC) EXPONENTIAL

$$\mathrm{Exp} : \big(tK_0(\mathrm{Var})[[t]], +\big) \xrightarrow{\ \sim\ } \big(1 + tK_0(\mathrm{Var})[[t]], \cdot\big)$$

$$\sum_{n>0} A_n t^n \longmapsto \prod_{n>0} (1-t^n)^{-A_n}$$

<u>Remark</u> $\mathrm{Exp}(t) = (1-t)^{-1} = 1+t+t^2+\cdots$ so $\mathrm{Exp}(-t) = 1-t \neq \mathrm{Exp}(t)|_{t\to -t}$

<u>Remark</u> $Y$ variety $\rightsquigarrow$ $\zeta_Y(t) := 1 + \sum_{n>0} [\mathrm{Sym}^n Y] t^n$ Kapranov's MOTIVIC ZETA FUNCTION.

The power structure is DETERMINED by $(1-t)^{-[M]} = \zeta_M(t)$. We are saying

$$\mathrm{Exp}\, A(t) = \prod_{n>0} \zeta_{A_n}(t^n)$$

# Hilbert series

$$Y \text{ variety} \rightsquigarrow Z_Y(t) := \sum_{n\geq 0} [\mathrm{Hilb}^n Y]\cdot t^n \in 1 + tK_0(\mathrm{Var})[[t]]$$

- $C$ smooth curve: $Z_C(t) = \zeta_C(t) = (1-t)^{-[C]} = \mathrm{Exp}([C]t)$
- $S$ smooth surface: $Z_S(t) = \prod_{n>0}(1-\mathbb{L}^{n-1}t^n)^{-[S]} = \mathrm{Exp}\left([S]\cdot\sum_{n>0}\mathbb{L}^{n-1}t^n\right) = \mathrm{Exp}\left(\frac{[S]t}{1-\mathbb{L}t}\right)$ 2001 Göttsche
- $Y$ smooth, $\dim Y = d$. $Z_Y(t) = \left(\sum_{n\geq 0}[\mathrm{Hilb}^n(\mathbb{A}^d)_0]\cdot t^n\right)^{[Y]}$ Gusein-Zade, Luengo, Melle-Hernández 2004

punctual Hilbert scheme ($Z \subset \mathbb{A}^d$, $\mathrm{Supp}\, Z = \{0\}$)

# Quot series

$F$ locally free, $\operatorname{rk} F = r$

$\downarrow$

$Y$ smooth quasiproj. $\dim Y = d$

$$\rightsquigarrow Q_F(t) := \sum_{n\geq 0} [\mathrm{Quot}_Y(F,n)]\, t^n \in K_0(\mathrm{Var})[\![t]\!]$$

$\{ F \twoheadrightarrow T \mid \dim(T) = 0,\ \chi(T) = n \}$

- $\dim Y = 1 \implies Q_F(t) = \mathrm{Exp}\left([Y \times \mathbb{P}^{r-1}]\, t\right)$ (Bifet: $\mathcal{O}_Y^{\oplus r}$) Bagnarol–Fantechi–Perroni 2020

- $\dim Y = 2 \implies Q_F(t) = \mathrm{Exp}\left(\dfrac{[Y \times \mathbb{P}^{r-1}]\, t}{1 - \mathbb{L}^r t}\right)$ Mozgovoy 2020

- $Q_F(t) = \left( \sum_{n\geq 0} [\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_0]\, t^n \right)^{[Y]}$ R. 2020

What should one do when no more "actual" invariants appear to be computable?

Compute virtual invariants!!

# THE VIRTUAL MOTIVE OF A CRITICAL LOCUS

$U$ smooth of dim $d$, $f \in \Gamma(U) \rightsquigarrow X = \mathrm{crit}(f) = Z(df) \subset U$ supports the complex $\Phi_f \in \mathrm{Perv}(X)$. Motivic version $\varphi_f \in K_0^{\hat{\mu}}(\mathrm{Var})$ defined by Denef - Loeser.

$$[X]_{vir} := -\mathbb{L}^{-\frac{d}{2}} \cdot \varphi_f \in \mathcal{M}^{\hat{\mu}} := K_0^{\hat{\mu}}(\mathrm{Var})[\mathbb{L}^{-\frac{1}{2}}]$$

definition due to Behrend - Bryan - Szendrői

e.g. $f = 0 \rightsquigarrow [U]_{vir} = \mathbb{L}^{-\frac{d}{2}}[U] \in \mathcal{M}$.

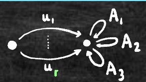
$$\begin{array}{ccc} \mathcal{M}^{\hat{\mu}} & \ni & [X]_{vir} \\ {\scriptstyle \mathbb{L}^{-\frac{1}{2}} \to -1}\ \Big\downarrow \chi & & \Big\downarrow \\ \mathbb{Z} & \ni & \chi_{vir}(X) := \chi_c(X, \Phi_f) \end{array}$$

this is DT(X) when X = moduli space of sheaves on a CY3. Then $[X]_{vir}$ is called motivic DT invariant.

## LOCAL RANK $r$ DT THEORY: $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$ IS A CRITICAL LOCUS

Fix $r, n$. Consider the quiver

$Q_3^r$ :




with potential
$W = A_1[A_2, A_3]$

$$\mathrm{Rep}_{(1,n)}(Q_3^r) = \mathrm{End}(\mathbb{C}^n)^{\oplus 3} \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n)^{\oplus r} \xrightarrow{f_{r,n}} \mathbb{A}^1$$

$$(A_1, A_2, A_3; u_1, \ldots, u_r) \longmapsto \mathrm{Tr}\, A_1[A_2, A_3]$$

" $f_{r,n} = \mathrm{Tr}\, W$ "

$GL_n$ - equivariant

$\Downarrow$

descends to $\mathrm{ncQuot}_r^n \xrightarrow{f_{r,n}} \mathbb{A}^1$

$\bigcup$ open

$$GL_n \overset{\text{FREE}}{\curvearrowright} \mathcal{U}_{r,n} = \left\{ (A_1, A_2, A_3; u_1, \ldots, u_r) \;\middle|\; \mathrm{Span}_{\mathbb{C}} \left\{ A_1^{a_1} A_2^{a_2} A_3^{a_3} \cdot u_j(1) \;\middle|\; \begin{array}{c} a_i \geq 0 \\ 1 \leq j \leq r \end{array} \right\} = \mathbb{C}^n \right\}$$

GIT quotient $\downarrow$

$$\mathrm{ncQuot}_r^n := \mathcal{U}_{r,n} / GL_n$$

SMOOTH OF DIM $2n^2 + rn$

**THEOREM** [Szendrői $r = 1$, Beentjes – R. any $r$]

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \xrightarrow{\sim} \mathrm{crit}(f_{r,n}) \subset \mathrm{ncQuot}_r^n$$

$$Z_r(t) := \sum_{n\geq 0} [\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)]_{\mathrm{vir}} \cdot t^n \in \mathcal{M}[[t]] \subset \mathcal{M}^{\hat{\mu}}[[t]]$$

- $$Z_1(t) = \prod_{m\geq 1} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{k+2-\frac{m}{2}} t^m\right)^{-1}$$ 2013 Behrend–Bryan–Szendrői

- $$Z_r(t) = \prod_{m\geq 1} \prod_{k=0}^{rm-1} \left(1 - \mathbb{L}^{k+2-\frac{rm}{2}} t^m\right)^{-1} = \prod_{1\leq i\leq r} Z_1\left(-\mathbb{L}^{\frac{-r-1}{2}+i} \cdot t\right)$$ 2015 Cazzaniga, 2017 R., Cazzaniga–Ralaivaosaona–R 2020 VIA MOTIVIC WALL-CROSSING

$\chi$

$$\sum_{n\geq 0} \chi_{\mathrm{vir}}(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)) \cdot t^n = \prod_{m\geq 1} (1-(-1)^{rm} t^m)^{-rm}$$

# GLOBAL THEORY

$Y$ any smooth 3-fold, $F$ any locally free sheaf of rank $r$.

R. one can define $[\mathrm{Quot}_Y(F,n)]_{\mathrm{vir}}$ via the local model

and one has

$$\sum_{n\geq 0}[\mathrm{Quot}_Y(F,n)]_{\mathrm{vir}}\cdot((-1)^r t)^n = \mathrm{Exp}\left((-1)^r t\,[Y\times \mathbb{P}^{r-1}]_{\mathrm{vir}}\cdot \mathrm{Exp}\left((-\mathbb{L}^{-\frac{1}{2}})^r t + (-\mathbb{L}^{\frac{1}{2}})^r t\right)\right)$$

# SANITY CHECK

$Y$ projective CY3 $\Rightarrow$ the above series REALLY

computes MOTIVIC DT INVARIANTS :

$$\sum_{n\geq 0} \chi\left[\mathrm{Quot}_Y(F,n)\right]_{\mathrm{vir}} \cdot t^n = \prod_{m\geq 1} \left(1-(-1)^{rm} t^m\right)^{-rm\chi(Y)}$$

$$\| \; \mathsf{R}.$$

enumerative DT

partition function

# Motivic measures $K_0(\mathrm{Var}) \to R$

$$
\begin{tikzcd}
K_0(\mathrm{Var}) \arrow[d, "\chi_h"] \arrow[dd, bend right, "E"'] \arrow[ddd, bend right=50, "P"'] \arrow[dddd, bend right=60, "\chi"'] & \ni & {[Y]},\ Y \text{ smooth projective} \arrow[d, mapsto] & \mathbb{L} = [\mathbb{P}^1]-[pt] \arrow[dd, mapsto] \\
K_0(HS) \arrow[d] & \ni & \sum_i (-1)^i [H^i(Y,\mathbb{Q})] \arrow[d, mapsto] & \\
\mathbb{Z}[u,v] \arrow[d, "{u,v \mapsto -t^{\frac{1}{2}}}"] & \ni & \sum_{p,q\geq 0} h^{p,q}(Y)(-u)^p(-v)^q \arrow[d, mapsto] & uv \arrow[d, mapsto] \\
\mathbb{Z}[t^{\frac{1}{2}}] \arrow[d] & \ni & \sum_{n\geq 0}(-1)^n \dim H^n(Y,\mathbb{Q})\cdot t^{\frac{n}{2}} \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\mathbb{Z} & \ni & \chi(Y) & 1
\end{tikzcd}
$$

$\chi_h, E, P, \chi$

are homomorphisms of **rings with power structure**

i.e.

$$\varphi\left((1-t)^{-[Y]}\right) = (1-t)^{-\varphi[Y]}$$

$\rightsquigarrow$ nice formulae

# Examples

power structure on $\mathbb{Z}[u,v]$ given by $(1-t)^{-\sum_{ij} p_{ij} u^i v^j} := \prod_{i,j} (1-u^i v^j t)^{-p_{ij}}$

$$\sum_{n\geq 0} E(\mathrm{Hilb}^n \mathbb{A}^2; u,v)\cdot t^n = E\left(\prod_{m\geq 1} (1-\mathbb{L}^{m-1} t^m)^{-\mathbb{L}^2}\right) = \prod_{m\geq 1} (1-(uv)^{m-1} t^m)^{-u^2v^2} = \prod_{m\geq 1} (1-(uv)^{m+1} t^m)^{-1}$$

$$\sum_{n\geq 0} h\left(H_c\left(\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n), \phi_{f_{r,n}}\right); x,y,z\right)\cdot (xyz^2)^{-n^2-rn}\cdot t^n$$

Davison $r=1$ $\quad \|$ $\quad$ R. $r$

$$\prod_{m\geq 1} \prod_{k=0}^{rm-1} \left(1-(xyz^2)^{k+2-rm} t^m\right)^{-1}$$

Hodge series of vanishing cycle cohomology for Quot scheme of $\mathbb{A}^3$ (Hodge-theoretic DT)

Thank you !