# Higher rank K-theoretic Donaldson–Thomas theory of points

with Nadir Fasola & Sergej Monavari
(Trieste) (Utrecht)

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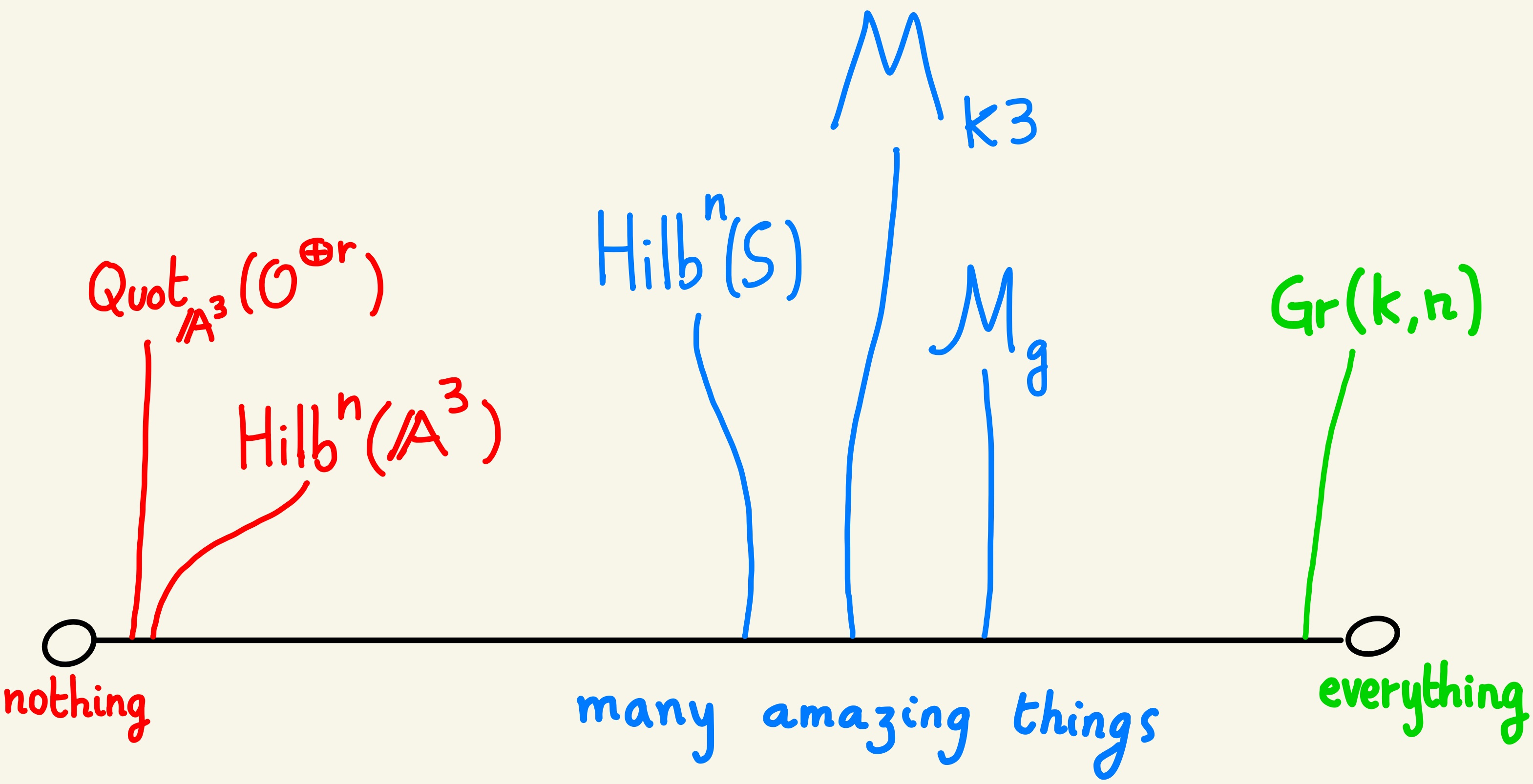
# WHAT WE KNOW ABOUT SOME MODULI SPACES




DT theory is one of the realms of Enumerative Geometry where objects we totally don't understand geometrically reveal amazing properties.

# Beyond numbers

$$\begin{tikzcd} & H^i_c(M,\mathbb{Q}) \arrow[d] & \text{vector space (Hodge structure)} \arrow[d, "\dim"] \\ M \arrow[r, squiggly] & b^i(M) \in & \mathbb{Z}_{\geq 0} \end{tikzcd}$$

DT theory has several natural refinements: we will see the K-theoretic refinement.

# DT theory: classical context

$X$ : smooth, projective Calabi–Yau 3-fold over $\mathbb{C}$.

$\longleftrightarrow$ $\Lambda^3 \Omega_X \cong \mathcal{O}_X$

$\gamma \in H^*(X)$

$\rightsquigarrow$ $M_X(\gamma)$ moduli space of sheaves with Chern character $\gamma$.

$\rightsquigarrow$ $DT(X, \gamma) \in \mathbb{Z}$

"DT invariant"

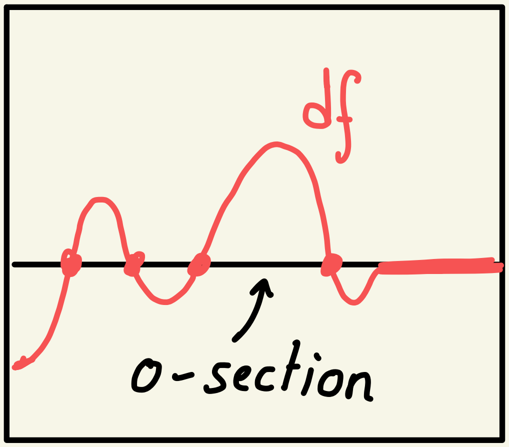
$\|$

"deformation invariant analogue of Euler characteristic $e(M_X(\gamma))$"

# KEY FACT on $M = M_X(\gamma)$

$$M \overset{\text{locally}}{\cong} \{df = 0\} \subset U, \text{ for } U \text{ a smooth scheme, } U \xrightarrow{f} \mathbb{C} \text{ a function.}$$

$\Omega_U$




"CRITICAL LOCUS"

$\{df = 0\}$ is virtually 0-dimensional

There is hope to count!

# WHAT IS SPECIAL ABOUT $Z = \{df = 0\}$ ?

(1) $\mathbb{H}^i_c(Z, \Phi_f)$

perverse sheaf of vanishing cycles

$$\mathbb{H}^i_c(Z, \Phi_f) \xrightarrow{\ e\ } e_{vir}(Z) \in \mathbb{Z}$$

$$e_{vir}(Z) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} \mathbb{H}^i_c(Z, \Phi_f)$$

computes DT invariant when $Z = M_X(\gamma)$

(2) $Z = \{df = 0\}$ has a canonical

SYMMETRIC OBSTRUCTION THEORY :

$$\begin{tikzcd}
\mathbb{E}_{\mathrm{crit}} \arrow[d, "\varphi"] & = \Big[ \overset{-1}{T_U|_Z} \arrow[r, "\mathrm{Hess}(f)"] \arrow[d, "(df)^\vee|_Z"] & \overset{0}{\Omega_U|_Z} \Big] \arrow[d, equal] \\
\mathbb{L}_Z & = \Big[ \mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] & \Omega_U|_Z \Big]
\end{tikzcd}$$

truncated cotangent complex

$\mathcal{I} \subset \mathcal{O}_U$ ideal sheaf of $Z \subset U$.

$$\left( \begin{array}{l} df : \mathcal{O}_U \to \Omega_U \\ (df)^\vee : T_U \twoheadrightarrow \mathcal{I} \subset \mathcal{O}_U \end{array} \right)$$

$\varphi$ induces:

(i) a VIRTUAL CLASS $[Z]^{vir} \in A_\circ Z$

(ii) a VIRTUAL STRUCTURE SHEAF

$$\mathcal{O}_Z^{vir} \in K_\circ(Z).$$

**Remark:**

$$K_{vir} := \det \mathbb{E}_{crit} = K_u\big|_Z^{\otimes 2}$$

has a canonical square root.

# ACTION STARTS NOW

The main player in HIGHER RANK DT THEORY OF POINTS is the Quot scheme

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n).$$

the simplest CY3...

Its points are short exact sequences

$$0 \to S \to \mathcal{O}^{\oplus r} \to T \to 0$$

where $\dim T = 0$, $\chi(T) = n$.

# Facts

local model for 0-dim DT theory.

(1) $r=1 \rightsquigarrow$ get $\mathrm{Hilb}^n(\mathbb{A}^3)$.

(2) $\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \overset{\text{globally}}{=} \{df = 0\}$.

[Beentjes-R 2018]
[Szendrői, r = 1]

$\Downarrow$

(3) Have $[\cdot]^{\mathrm{vir}}$, $\mathcal{O}^{\mathrm{vir}}$, $K^{\frac{1}{2}}_{\mathrm{vir}}$.

(4) There is a $\mathbb{T}$-action on the Quot scheme

$$\text{Torus } \mathbb{T} = \overset{T_1}{\overset{\parallel}{(\mathbb{C}^\times)^3}} \times \overset{T_2}{\overset{\parallel}{(\mathbb{C}^\times)^r}}$$

moves the support of $T \twoheadleftarrow \mathcal{O}^{\oplus r}$ via

rescales $\mathcal{O}^{\oplus r}$

$$(t_1, t_2, t_3) \cdot (x_1, x_2, x_3) = (t_1 x_1, t_2 x_2, t_3 x_3)$$

$$\boxed{\mathbb{T}\text{-action on } \quad Q_{r,n} = \mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)}$$

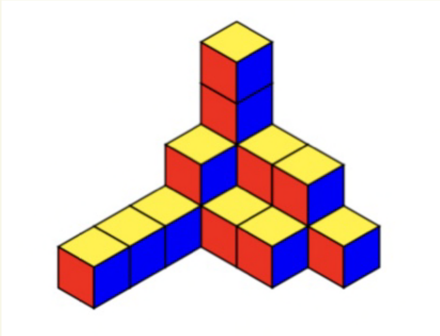
**Lemma** $\mathbb{T}$ - fixed locus is:

$$Q_{r,n}^{\mathbb{T}} = \coprod_{n_1+\cdots+n_r=n} \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathbb{A}^3)^{T_1}$$

$T_1 = (\mathbb{C}^\times)^3$ - fixed locus of $\mathrm{Hilb}^k(\mathbb{A}^3)$ is reduced, finite, isolated, indexed by

MONOMIAL IDEALS OF COLENGTH $k$ $\longleftrightarrow$ PLANE PARTITIONS OF SIZE $k$

$I_\pi \subset \mathbb{C}[x_1, x_2, x_3] \longleftrightarrow \pi$



A plane partition $\pi$ of size $|\pi| = 16$.

$$S \hookrightarrow \mathcal{O}^{\oplus r} \twoheadrightarrow T$$

$$Q^{\mathbb{T}}_{r,n} \ni [S] \leftrightarrow S = \bigoplus_{i=1}^{r} I_{\pi_i}$$

$$Q^{\mathbb{T}}_{r,n} \cong \Big\{ r\text{-colored plane partitions } \overline{\pi} = (\pi_1, \dots, \pi_r) \text{ of size } |\overline{\pi}| = \sum_{i=1}^{r} |\pi_i| = n \Big\}$$

# K-THEORETIC INVARIANTS

$\mathbb{E} \xrightarrow{\varphi} \mathbb{L}_Y$ p.o.t. on a scheme $Y \rightsquigarrow \mathcal{O}_Y^{vir} \in K_0^T(Y)$

$Y \xrightarrow{p} pt$ proper $\rightsquigarrow \chi(Y,-) = p_*: K_0^T(Y) \longrightarrow K_0^T(pt)$

$$\boxed{\chi^{vir}(Y,V) = \chi(Y, V \otimes \mathcal{O}_Y^{vir}).}$$

Important characters:

$\mathbb{E} = [E^{-1} \to E^0]$

$T_Y^{vir} = \mathbb{E}^\vee = E_0 - E_1$ virtual tangent space

$$N^{vir} = N^{vir}_{Y^T/Y} = T_Y^{vir}\Big|_{Y^T}^{mov} \in K_0^T(Y^T).$$

virtual normal bundle

# Virtual localisation

$Y$ has a $T$-action $\Rightarrow$ $Y^T$ has its own $\mathcal{O}^{vir}$

$$\chi^{vir}(Y,V) = \chi^{vir}\left(Y^T, \frac{V|_{Y^T}}{\Lambda^\bullet N^{vir,\vee}}\right)$$

$$\rotatebox{90}{$\in$} \quad K_0^T(pt)\left[(1-t^\mu)^{-1} \,\middle|\, \mu \in \widehat{T}\right]$$

If $Y$ is not proper, <u>DEFINE</u> $\chi^{vir}(Y,V)$ to be the RHS, provided that $Y^T$ is proper.

# Definition of higher rank DT invariants of $\mathbb{A}^3$

- $\widehat{\mathcal{O}}^{\mathrm{vir}} = \mathcal{O}^{\mathrm{vir}} \otimes K_{\mathrm{vir}}^{\frac{1}{2}}$ on each $Q_{r,n}$
- $\mathrm{tr} : K_0^{\mathbb{T}}(\mathrm{pt}) \xrightarrow{\sim} \mathbb{Z}[t^{\mu} \mid \mu \in \widehat{\mathbb{T}}]$

$$\mathrm{DT}^K_{r,n} = \chi(Q_{r,n}, \widehat{\mathcal{O}}^{\mathrm{vir}})$$

$$= \sum_{[S]\in Q^{\mathbb{T}}_{r,n}} \mathrm{tr}\left(\frac{K_{\mathrm{vir}}^{1/2}}{\Lambda^{\bullet}(T_S^{\mathrm{vir}})^{\vee}}\right)$$

$$\cap$$

$$\mathbb{Z}(\!(t, (t_1t_2t_3)^{\frac{1}{2}}, w)\!).$$

$t_1, t_2, t_3$ $\quad$ $w_1, \ldots, w_r$

Form the generating function

$$\mathrm{DT}_r^K(\mathbb{A}^3, t, w, q) = \sum_{n \geq 0} \mathrm{DT}_{r,n}^K \cdot q^n .$$

Okounkov proved Nekrasov's conjecture:

$$\mathrm{DT}_1^K(\mathbb{A}^3, t_1, t_2, t_3, w_1, -q) = \mathrm{Exp}\left(\frac{1}{[\underline{t}^{1/2} q][\underline{t}^{1/2} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}\right)$$

- Exp = plethystic exponential
- $[x] = x^{1/2} - x^{-1/2}$. $\underline{t} = t_1 t_2 t_3$
- Note the independence on $w_1$

# Main Theorem

**Theorem** (Fasola-Monavari-R)

$$\mathrm{DT}_r^K(\mathbb{A}^3, t, w, (-1)^r q) = \mathrm{Exp}\, F_r(q, t_1, t_2, t_3)$$

- $$F_r := \frac{[t^r]}{[\underline{t}][\underline{t}^{r/2} q][\underline{t}^{r/2} q^{-1}]} \frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[t_1][t_2][t_3]}$$

  where $\underline{t} = t_1 t_2 t_3$

- This was conjectured by Awata-Kanno (2009) in string theory.
- Note the independence on $w = (w_1, \ldots, w_r)$

# Before we go on

(•) A proof of the Awata–Kanno conjecture was also announced by Noah Arbesfeld and Yasha Kononov.

(•) A "10 years later" review on this conjecture was arxived by Kanno last week.

(•) One more remark on the Physics literature....

.... The plethystic formula for $DT_r^K(\mathbb{A}^3, t, w, (-1)^r q)$ is equivalent to

$$\boxed{\begin{gathered} DT_r^K(\mathbb{A}^3, t, (-1)^r q) \\ \| \\ \prod_{i=1}^{r} DT_1^K\left(\mathbb{A}^3, -q\,\underline{t}^{\frac{-r-1}{2}+i}, t\right) \end{gathered}}$$

i.e. the rank 1 theory determines the rank r theory.

This formula appeared in the work of $\boxed{\text{Nekrasov–Piazzalunga}}$ as a limit of (conjectural) 4-fold identities.

# Ingredients in the proof

(1) explicit formula for $T_S^{vir}$, $[S] \in Q_{r,n}^{\mathrm{T}}$.

$$T_S^{vir} = \mathbb{E}_{crit}^{\vee}\big|_{[S]} \in K_0^{\mathrm{T}}(pt)$$

! (2) $DT_r^K(\mathbb{A}^3, t, w, q)$ does not depend on $w$

(3) Evaluate $DT_r^K(\mathbb{A}^3, t, w, q) = \sum_{\bar{\pi}} q^{|\bar{\pi}|} \prod_{i,j=1}^{r} [-V_{ij}]$

by setting $w_i = L^i$ and computing the limit for $L \to \infty$.

$\rightsquigarrow$ This yields the product formula.

Conclude by properties of Exp.

these are "vertex terms" arising from localisation.

$$T_S^{vir} = \chi(\mathcal{O}^{\oplus r}, \mathcal{O}^{\oplus r}) - \chi(S, S) = \sum_{1 \leq i,j \leq r} V_{ij}$$

determines the HIGHER RANK VERTEX, where

$$V_{ij} = w_i^{-1} w_j \left( Q_j - \frac{\overline{Q}_i}{t_1 t_2 t_3} + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} Q_j \overline{Q}_i \right)$$

$$Q_i = tr_{\mathcal{O}/I_{\pi_i}} = \sum_{\square \in \pi_i} t^{\square} .$$

rank r version of MNOP, including the triviality of the $\mathbb{T}$-fixed p.o.t. on $Quot_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)^{\mathbb{T}}$.

# COHOMOLOGICAL (rank r) DT INVARIANTS

$$\mathrm{DT}^{\mathrm{coh}}_{r,n} = \int_{[\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)]^{\mathrm{vir}}} 1$$

$s_i = c_1^{\mathbb{T}}(t_i)$

$v_j = c_1^{\mathbb{T}}(w_j)$

localisation

$$= \sum_{[S]\ \mathbb{T}\text{-fixed}} e^{\mathbb{T}}(-T_S^{\mathrm{vir}}) \in \mathbb{Q}(s, v)$$

$$\boxed{\mathrm{DT}^{\mathrm{coh}}_r(q) := \sum_{n \geq 0} \mathrm{DT}^{\mathrm{coh}}_{r,n} \cdot q^n}$$

true if $r=1$ [MNOP]

SZABO'S CONJECTURE

$$\mathrm{DT}^{\mathrm{coh}}_r(q) = M((-1)^r q)^{-r\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3}}$$

$$M(q) = \sum_{\pi} q^{|\pi|} = \prod_{m \geq 1} (1-q^m)^{-m}$$

**THEOREM (Fasola-Monavari-R)**

Szabo's conjecture is true.

# "proof":

(1) $\mathrm{DT}_r^{coh}(q) = \lim_{b \to 0} \mathrm{DT}_r^{K}(\mathbb{A}^3, e^{bs}, e^{bv}, q)$.

(2) $\mathrm{DT}_r^{coh}(q)$ does not depend on $e^{\mathbb{T}}(W)$.

(3) Compute $\lim_{b \to 0} \operatorname{Exp} F_r(q, t_1^b, t_2^b, t_3^b)$.

# FUTURE of DT THEORY OF POINTS

We propose a definition of virtual chiral elliptic genus.
There is currently no guessed formula for the partition function
(not even in Physics ! )

?

↓

✓ K-theoretic

↓

✓ Cohomological

↓

enumerative

one formula that still awaits proof is :

$$\sum_{n \geq 0} q^n \int_{[\mathrm{Quot}_X(F,n)]^{vir}} 1 = M((-1)^r q)^{r \cdot \int_X c_3 - c_1 c_2}$$

Thank you !!