SHEAVES on K3 SURFACES

[Huybrechts, Ch. 10]

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Overview

(0) Moduli of sheaves: what to expect

(1) General theory

(2) On K3 surfaces

- (2.1) tangent space to moduli
- (2.2) Symplectic structure
- (2.3) Examples

Notions of "moduli space"

Have: a functor $Sch_{\mathbb{C}}^{op} \xrightarrow{\mathcal{M}} Sets$

Want: a scheme M relating nicely to $\mathcal{M}$

(1)
$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\eta"] \arrow[dr, "\forall"'] & \mathrm{Hom}_{Sch_{\mathbb{C}}}(-, M) \arrow[d, dotted, "\exists!"] \\
& \mathrm{Hom}_{Sch_{\mathbb{C}}}(-, N)
\end{tikzcd}$$

η universal natural transformation

$\updownarrow$

"M is a moduli space for $\mathcal{M}$"

(2) M is a coarse moduli space:

have η, and $\mathcal{M}(\mathbb{C}) \xrightarrow{\sim} M(\mathbb{C})$.

(3) M is a fine moduli space:

$$\mathcal{M} \xrightarrow[\cong]{\eta} \mathrm{Hom}_{Sch_{\mathbb{C}}}(-, M)$$

(0) WHAT TO EXPECT

Say we want to parametrise rank r bundles on $\mathbb{P}^1$

$r = 1 \rightsquigarrow$ Picard scheme :)

$r = 2 \rightsquigarrow$ two problems
- FINITE TYPE (i)
- SEPARATEDNESS (ii)

(i)

$$\underbrace{M_{\mathbb{P}^1}(\overset{r}{2}, \overset{c_1}{0})}_{} \supseteq \{\mathcal{O}(-n) \oplus \mathcal{O}(n)\}_{n>0} \qquad h^0 = n+1$$

$$\{E \mid h^0(E) \geq n\} \overset{\text{closed}}{\hookrightarrow} M_{\mathbb{P}^1}(2,0)$$

$\implies$ get infinite descending chain of closed subschemes

$\Downarrow$

$M_{\mathbb{P}^1}(2,0)$ would not be finite type

(ii) Again $r=2$. $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) \cong \mathbb{C}$

$$[\mathcal{O}(-1) \hookrightarrow E_\lambda \twoheadrightarrow \mathcal{O}(1)] \longleftrightarrow \lambda$$

$$\begin{array}{ccccc} \mathcal{O}(-1)\oplus\mathcal{O}(1) & & \mathcal{E} & & E_\lambda \cong \mathcal{O}^{\oplus 2} \\ | & & | & & | \\ \{0\}\times\mathbb{P}^1 & \subset & \mathbb{A}^1\times\mathbb{P}^1 & \supset & \{\lambda\}\times\mathbb{P}^1 \end{array}$$

base of the family

If we had a fine M, $\mathcal{E}$ would give us $\mathbb{A}^1 \xrightarrow{\varphi} M$

But... $\varphi(0)$
- $[\mathcal{O}(-1)\oplus\mathcal{O}(1)] \Rightarrow M$ non-separated
- $[\mathcal{O}^{\oplus 2}] \Rightarrow M$ not fine!

To solve both problems at the same time, introduce **STABILITY**. For curves, this is Mumford's μ-stability. In general,

GIESEKER STABILITY

(1) General theory

X : smooth projective variety over $\mathbb{C}$ any $k=\bar{k}\ldots$

H : fixed polarisation (ample divisor)

Hilbert polynomial of E w.r.t. H

$$E \in \mathrm{Coh}_X \rightsquigarrow P(E,m) = \chi(E \otimes \mathcal{O}_X(mH))$$

GRR

$$= \int_X \mathrm{ch}(E).\,\mathrm{ch}(\mathcal{O}_X(mH)).\underbrace{\mathrm{Td}(X)}_{1+\frac{c_1(X)}{2}+\frac{c_1(X)^2+c_2(X)}{12}+\cdots}$$

EXAMPLE: X SURFACE

$$\mathrm{ch}(E) = (r, c_1, \overset{\mathrm{ch}_2(E)}{\overset{\parallel}{n}})$$

$$P(E,m) = \int_X (r, c_1, n)\cdot\left(1, mH, \frac{m^2H^2}{2}\right)\cdot\left(1, \frac{c_1(X)}{2}, \frac{c_1(X)^2+c_2(X)}{12}\right)$$

$$= \underbrace{r\frac{H^2}{2}m^2 + c_1Hm + n}\ + \underbrace{\frac{c_1c_1(X)}{2} + r\frac{Hc_1(X)}{2}m}\ + r\frac{c_1(X)^2+c_2(X)}{12}$$

$$\boxed{\begin{array}{l}\text{EXAMPLE: } X \ \mathrm{K3} \rightsquigarrow c_2(X) = 24 \\[2mm] P(E,m) = r\dfrac{H^2}{2}m^2 + c_1Hm + n + 2r\end{array}}$$

write $P(E,m) = \sum_{0 \leq i \leq \dim E} \overbrace{\alpha_i(E)}^{\in\, \mathbb{Z}} \frac{m^i}{i!}$

$$p(E,m) = P(E,m)/\alpha_d(E), \quad d = \dim E$$

reduced Hilbert polynomial

$F \subseteq E \Rightarrow \dim F = \dim E$

torsion free $\Rightarrow$

E is STABLE if it is pure, and

$$0 \neq F \subsetneq E \Longrightarrow p(F,m) < p(E,m), \quad m \gg 0$$

$\leqslant \rightsquigarrow E$ SEMISTABLE

Examples (X surface)

$d = 0$. $p(E, m) = 1$. E pure, semistable.

$$E \text{ stable} \iff E \cong k(x),\ x \in X.$$

$d = 1$: $E = \iota_* F$, F locally free on $C \xhookrightarrow{\iota} X$

$$\mu\text{-stability of } F \iff \text{stability of } E$$

$d = 2$: Assume X is a K3, so $c_2(X) = 24$

$$P(E, m) = \alpha_0(E) + \alpha_1(E)\, m + \alpha_2(E)\, \frac{m^2}{2}$$

$$= 2r_E + ch_2(E) + c_1(E)\cdot H \cdot m + r_E H^2\, \frac{m^2}{2}$$

$$p(E, m) = \frac{\alpha_0(E)}{r_E H^2} + \frac{c_1(E)\cdot H}{r_E H^2}\, m + \cancel{\frac{m^2}{2}}$$

? V

$$p(F, m) = \frac{\alpha_0(F)}{r_F H^2} + \frac{c_1(F)\cdot H}{r_F H^2}\, m + \cancel{\frac{m^2}{2}}$$

E torsion free

$\cup\!\!\!\!\!\#$

F

$$\boxed{\begin{array}{l} \dfrac{c_1(E)\cdot H}{r_E H^2} = \dfrac{c_1(F)\cdot H}{r_F H^2} \implies \dfrac{\alpha_0(F)}{r_F H^2} < \dfrac{\alpha_0(E)}{r_E H^2}, \text{ or} \\[2ex] \dfrac{c_1(F)\cdot H}{r_F H^2} < \dfrac{c_1(E)\cdot H}{r_E H^2} \qquad \text{STABILITY OF } E \end{array}}$$

The functor

Fix: X, H, P

$$\mathcal{S}ch_{\mathbb{C}}^{op} \xrightarrow{\mathcal{M}} \mathcal{S}ets \qquad \mathcal{M}^{st} \subset \mathcal{M}$$

$$B \longmapsto \left\{ \mathcal{E} \in Coh(X \times B) \;\middle|\; \begin{array}{c} \mathcal{E} \text{ is } B\text{-flat}, \mathcal{E}_b \text{ is} \\ \text{semistable}, P(\mathcal{E}_b, m) = P \;\; \forall b \end{array} \right\} \Big/ \sim$$

$$\mathcal{E} \sim \mathcal{F} \text{ if } \mathcal{E} \cong \mathcal{F} \otimes \pi_B^* \mathcal{L}, \quad X \times B \xrightarrow{\pi_B} B$$

(weakest notion...)

$$\boxed{\text{THEOREM } \; \mathcal{M} \text{ has a projective moduli space } M.}$$

$$M^{st} \overset{\text{open}}{\subset} M \qquad \text{stable locus}$$

<u>Example (d=0)</u> $P \equiv n \Rightarrow M = \mathrm{Sym}^n X = X^n / \mathfrak{S}_n$.

Construction via G.I.T.

G.I.T. - semistability

$\parallel\!\mid$

quotient sheaf is

Gieseker - semistable

$$
\begin{tikzcd}
U^{st} \arrow[r, hook, "\text{open}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[r, hook, "\text{open}"] \arrow[d, "\text{G.I.T. quotient}"] & \mathrm{Quot}\left(\mathcal{O}_X(-m)^{\oplus P(m)}, P\right) \\
M^{st} \arrow[r, hook, "\text{open}"] & U/PGL(P(m)) = M &
\end{tikzcd}
$$

You can't always get what you want

$$M(\mathbb{C}) = \left\{ \begin{array}{l} \text{S-equivalence classes of} \\ \text{semistables } E \text{ with } P(E) = P \end{array} \right\}$$

E semistable $\rightsquigarrow 0 \subset E_0 \subset \cdots \subset E_s = E$ Jordan-Hölder filtration

E_i / E_{i-1} stable with reduced Hilbert pol. $p(E, m)$.

$$E \underset{S}{\sim} F \iff \bigoplus E_i / E_{i-1} \cong \bigoplus F_i / F_{i-1}$$

So, if you find a STRICTLY SEMISTABLE sheaf, then $\mathcal{M}$ cannot be represented i.e. $\mathcal{M}$ is not fine.

Local structure of $\mathcal{M}$

Pretend M is fine. Then, if $[E] \in M$ corresponds to a stable sheaf E, one has

$$\underbrace{T_{[E]} M} = \mathrm{Hom}_{[E]}(\mathrm{Spec}\, \mathbb{C}[t]/t^2, M)$$

$$= \mathcal{M}(\mathrm{Spec}\, \mathbb{C}[t]/t^2) = \underline{\mathrm{Ext}^1(E,E)}.$$

↳ This happens even if M is not fine.

? $\Longrightarrow$ Smoothness

$$\mathrm{Ext}^2(E,E) = 0 \Rightarrow M \text{ smooth at } [E].$$

$\Uparrow$ ← now we exploit this on a K3 ...

$$\mathrm{Ext}^2(E,E) \xrightarrow{tr} H^2(\mathcal{O}_X) \text{ injective}$$

Trace map

$$\boxed{\begin{tikzcd} \mathcal{O}_X \arrow[r, "\mathrm{id}_E"] & R\mathcal{H}om(E,E) \arrow[l, bend left, dotted, "\mathrm{tr}"] \end{tikzcd} \qquad \mathrm{tr}\circ \mathrm{id}_E = \cdot\, \mathrm{rk}(E)}$$

(Assume $\mathrm{rk}(E) \neq 0$)

$$\mathrm{tr} = h^i(\mathrm{tr}) : \mathrm{Ext}^i(E,E) \longrightarrow H^i(\mathcal{O}_X)$$

$$\left(\text{e.g. } h^1(\mathrm{tr}) = \text{tangent map to } \begin{array}{ccc} M^{st} & \xrightarrow{\det} & \mathrm{Pic}_X \\ [E] & \longmapsto & [\det E] \end{array} \right)$$

X K3 surface. Then

- $r=1$: $0 = H^1(\mathcal{O}_X) = \mathrm{Ext}^1(L,L) \Rightarrow \mathrm{Pic}_X$ reduced, isolated.

- $r>1$: $\mathrm{Ext}^2(E,E) \cong \mathrm{End}(E)^\vee \cong \mathbb{C}$ (E stable)

$$\begin{array}{c} \mathrm{Ext}^2(E,E) \\ \downarrow \mathrm{tr} \\ \mathbb{C} \cong H^2(\mathcal{O}_X) \end{array} \quad \rightsquigarrow \quad \mathrm{tr}^\vee : H^0(\mathcal{O}_X) \xrightarrow[\sim]{f \mapsto f\cdot \mathrm{id}} \mathrm{End}(E)$$

So $\mathrm{tr} \neq 0 \Longrightarrow \boxed{M^{st} \text{ smooth of dim } \mathrm{Ext}^1(E,E)}$

(2) From now on, X is a K3 surface

$$\alpha, \beta \in H^*(X, \mathbb{Z})$$

$$\langle \alpha, \beta \rangle = \alpha_2 \cdot \beta_2 - \alpha_0 \cdot \beta_4 - \alpha_4 \cdot \beta_0$$

$\cap$ form

MUKAI PAIRING

$P \longleftrightarrow$ Mukai vector

$$v = v(E) = (\underset{\overset{\|}{ch_0}}{\mathrm{rk}\, E}, \underset{\overset{\|}{ch_1}}{c_1(E)}, ch_2(E) + \mathrm{rk}\, E) \in H^*(X, \mathbb{Z})$$

$$P(E, m) := \chi(E(m)) \underset{\Uparrow}{=} -\langle v(E), v(\mathcal{O}_X(mH)) \rangle$$

$$\chi(E, F) = -\langle v(E), v(F) \rangle$$

$$-\langle v(E), v(E) \rangle = \chi(E, E) = \sum_{i \geq 0} (-1)^i \operatorname{ext}^i(E, E)$$

$$= 2 - \operatorname{ext}^1(E, E)$$

$$\boxed{M^{st} \text{ smooth of dimension } 2 + \langle v, v \rangle \text{ (or } \emptyset)}$$

$$T_{M^{st}} \xrightarrow{\sim} \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E})$$

$\Big(\mathcal{E}$: ⚠ univ. sheaf on $X \times M^{st} \xrightarrow{p} M^{st}\Big)$

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proof

$$\begin{array}{ccc} At(\mathcal{E}) \in Ext^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1_{M^{st}}) \\ \downarrow \quad \text{Grothendieck duality along } p \\ \varphi \in \Gamma(M^{st}, \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}) \otimes \Omega^1_{M^{st}}) \end{array}$$

not enough...

$$T_{M^{st}} \xrightarrow{\varphi} \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}), \quad T_{[E]}M^{st} \xrightarrow[\varphi|_{[E]}]{\sim} Ext^1(E,E)$$

$$\begin{array}{c} U^{st} \\ \pi \downarrow \\ M^{st} \end{array} \qquad \begin{array}{c} \widetilde{\mathcal{E}} \\ | \\ X \times U^{st} \xrightarrow{\tilde{p}} U^{st} \end{array}$$

$\widetilde{\mathcal{E}}$ univ. quotient

$$\begin{tikzcd} T_{U^{st}} \arrow[r, "KS"] \arrow[d] & \mathcal{E}xt^1_{\tilde{p}}(\widetilde{\mathcal{E}},\widetilde{\mathcal{E}}) \\ \pi^* T_{M^{st}} \arrow[ur, "\sim", "\tilde{\varphi}"'] & \end{tikzcd}$$

$\tilde{\varphi}$ PGL-invariant via equivariance of $At_{\widetilde{\mathcal{E}}}$ $\Rightarrow$ it descends to φ, which is then also an isomorphism. ▨

COROLLARY: SYMPLECTIC STRUCTURE

$$T_{M^{st}} \times T_{M^{st}} \xrightarrow{\sim} \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}) \times \mathcal{E}xt^1_p(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{O}_{M^{st}}$$

relative Serre duality $\rightsquigarrow$ everywhere non-degenerate

$$\sigma \in \Gamma\left(M^{st}, \Omega^2_{M^{st}}\right)$$

(1) First observed by Mukai

(2) σ is closed [Huybrechts – Lehn, Ch. 10]

(3) Reasonable (but false) hope: realise $M^{st}(v)$ as new examples of IRREDUCIBLE SYMPLECTIC MANIFOLDS.

How to avoid strictly semistables

Fix $v = (r, \ell, s) \in H^*(X, \mathbb{Z})$ either $r > 0$ or $s \neq 0$ if $r = 0$

$$\left.\begin{array}{l} v \text{ primitive} \\ H \text{ generic} \end{array}\right\} \Longrightarrow M^{st}(v) = M(v)$$

smooth projective of dim. $\langle v, v \rangle + 2$, or $\emptyset$

$\langle v, v \rangle \neq -2$

(v not primitive $\Rightarrow$ $M(v)$ might be singular)

HAVE WE BEEN TALKING ABOUT $\emptyset$ ALL ALONG?

Theorem NO: $v = (r, \ell, s)$ fixed Mukai vector.

$$\left.\begin{array}{l} \langle v, v \rangle \geq -2, \text{ and} \\ r > 0 \text{ or } \ell \text{ ample} \end{array}\right\} \Longrightarrow M(v) \neq \emptyset.$$

EXAMPLES

$M^{st} \ni [E]$, $\mathrm{Ext}^1(E,E) = 0$ i.e. E rigid.

$$\langle v, v \rangle = -\chi(E,E) = -\mathrm{ext}^0(E,E) - \mathrm{ext}^2(E,E) = -2$$

$$\langle v, v \rangle = -2 \implies M^{st} = \begin{cases} \emptyset \\ \mathrm{Spec}\, \mathbb{C} \quad (\cong M(v)) \end{cases}$$

<u>THEOREM</u> (Mukai) If $\langle v, v \rangle = 0$ and $Y \subset M(v)^{st}$ is a complete component $\implies Y = M(v)^{st} = M(v)$.

either $\emptyset$, or
smooth, 2-dimensional

Exercise: E $\underbrace{\text{stable, rigid}}_{\text{not easy to find...}}$, $\mathrm{rk}(E) \geq 1 \implies E$ locally free

Corollary $v=(r,\ell,s)$ primitive, either $r\geq 1$ or $s\neq 0$, $\langle v,v\rangle=0$.

Then, for generic H, $M(v)$ is a K3 surface.

Proof. We know $M:=M^{st}=M(v)$ is a smooth projective surface, with $\sigma\in\Gamma(\Omega^2_M)$. Then $\omega_M\cong\mathcal{O}_M$.

Goal: $H^1(\mathcal{O}_M)=0$.

$\mathcal{E}$ univ. sheaf

$$\begin{tikzcd} & X\times M \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X & & M \end{tikzcd} \rightsquigarrow D^b(M)\xrightarrow[FM_{\mathcal{E}}]{\sim} D^b(X)$$

Bondal-Orlov criterion (surfaces with $\omega\cong\mathcal{O}$)

$$H^i(X,\mathcal{E}xt^j_q(\mathcal{E},\mathcal{E}))\Rightarrow \mathrm{Ext}^{i+j}(\mathcal{E},\mathcal{E})$$

$$H^i(M,\mathcal{E}xt^j_p(\mathcal{E},\mathcal{E}))\Rightarrow \mathrm{Ext}^{i+j}(\mathcal{E},\mathcal{E})$$

$$H^1(M,\mathcal{O}_M)\hookrightarrow \mathrm{Ext}^1(\mathcal{E},\mathcal{E})\hookleftarrow H^1(X,\mathcal{O}_X)=H^0(X,T_X)=0$$

$$T_X=\mathcal{E}xt^1_q(\mathcal{E},\mathcal{E})$$

$$\Rightarrow \mathrm{Ext}^1(\mathcal{E},\mathcal{E})=0\Rightarrow H^1(M,\mathcal{O}_M)=0 \qquad \blacksquare$$

Example $X \subset \mathbb{P}^3$ quartic. $v = (2, \mathcal{O}_X(-1), 1)$

$$\rightsquigarrow \quad \begin{array}{ccc} M(v) & \xrightarrow{\sim} & X \\ \cup & & \cup \\ [E] & \longmapsto & x \end{array} \qquad E \hookrightarrow \mathcal{O}_X^{\oplus 3} \twoheadrightarrow \mathcal{I}_x$$

Example $v = (1, 0, 1-n)$, $n \geq 0$.

$$\left\{ \begin{array}{l} \text{torsion free sheaves } E \\ \text{with } \mathrm{rk}(E) = 1,\ \det(E) = \mathcal{O}_X \end{array} \right\} = \left\{ \begin{array}{c} \text{ideal sheaves} \\ \mathcal{I}_Z \subset \mathcal{O}_X,\ \mathrm{codim}\, Z = 2 \end{array} \right\}$$

$$\rightsquigarrow \quad \begin{array}{ccccc} \mathrm{Hilb}^n(X) & \xrightarrow{\sim} & M(1,0,1-n) & = & M(1,0,1-n)^{st} \\ \downarrow & & \downarrow & & \\ \mathrm{Sym}^n X & \xrightarrow{\sim} & M(0,0,n) & \supset & M(0,0,n)^{st} \underset{n \geq 1}{=} \emptyset \end{array}$$

THEOREM $v = (r, \ell, s)$ primitive, $r \geq 1$ or $s \neq 0$, $\langle v, v \rangle \geq -2$.

Then, for generic H, $M(v)$ is an irreducible symplectic projective manifold, deformation equivalent to

$$\mathrm{Hilb}^n(X), \quad 2n = \langle v, v \rangle + 2.$$

[Huybrechts, Yoshioka]